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DETERMINANTS, INTEGRALITY AND NOETHER'S THEOREM FOR QUANTUM COMMUTATIVE ALGEBRAS

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In memory of Professor Shimshon Amitsur

ABSTRACT

The aim of this paper is to generalize Noether's theorem for finite groups acting on commutative algebras, to finite-dimensional triangular Hopf algebras acting on quantum commutative algebras. In the process we construct a non-commutative determinant function which yields an analogue of the Cayley–Hamilton theorem for certain endomorphisms.

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Let G be a finite group acting as automorphisms on a commutative k-algebra A, and A^G the subalgebra of G-invariants. An old result of E. Noether, based on the easy fact that A is integral over A^G, is that if A is k-affine then so is A^G. This result can be generalized using [DG] or directly using determinants as [F-S] to actions of any finite-dimensional cocommutative Hopf algebra H on a commutative algebra A (kG being an example of such an H). In a recent paper, it was shown by Zhu [Z] that cocommutativity of H can be replaced by semisimplicity. Failing this, however, the result might not hold.

The aim of this paper is to prove Noether's theorem for finite-dimensional semisimple triangular (H, R), but where A is no longer commutative in the usual sense, it is quantum commutative. If H is cocommutative, with $R = 1 \otimes 1$, then A is quantum commutative if and only if A is commutative in the usual sense. Our philosophy is that quantum-commutative H-module algebras with respect to (H, R) share many of the properties of commutative algebras acted upon by cocommutative H [CW]. The results in this paper give more evidence for this point of view. In the process we construct a non-commutative determinant function which yields an analogue of the Cayley–Hamilton theorem for certain endomorphisms. This determinant function is constructed for a wide class of Hopf algebras.

Specifically, let $(H, \langle\ |\ \rangle)$ be a cotriangular Hopf algebra over k; then the category whose objects (V, ρ) are right H-comodules is a symmetric monoidal category with a "twist" map Ψ induced from $\langle\ |\ \rangle$. That is:

$$\Psi(v \otimes w) = \sum \langle w_1 | v_1 \rangle w_0 \otimes v_0 \quad \text{(where } \rho(v) \text{ is written as } \textstyle\sum v_0 \otimes v_1\text{)}.$$

This twist map induces an action of the symmetric group on the ith-fold tensor $V^{\otimes i}$, which in turn gives rise to an appropriate Grassman algebra. Using these ideas we define in Definition 2.9 a determinant function which satisfies the following:

THEOREM 2.10: *Let $(H, \langle\ |\ \rangle)$ be a cotriangular Hopf algebra over k. Let A be a quantum commutative right H-comodule algebra, and V an n-dimensional right H-comodule so that $\sum \langle v_2 | v_1 \rangle v_0 = v$, for all $v \in V$. Assume* $\operatorname{Char} k = 0$ *or* $\operatorname{Char} k > n$. *Let $S, T \in \operatorname{End}(A \otimes V)$ be morphisms in ${}_A\mathcal{M}^H$. Then*

1. $\det(T) \in A^{\mathrm{co}H} \subset Z(A)$.
2. $\det(T)$ *is independent of the choice of basis of $\bigwedge_B^n(A \otimes V)$ as a left A-module.*
3. $\det(I) = 1$.

4. $\det(T \circ S) = \det(T)\det(S)$.

Dually, if (H, R) is a triangular Hopf algebra and A and V are the analogous objects in the category of left H-modules, we construct a determinant function and prove in:

THEOREM 2.19: *Let (H, R) be a triangular Hopf algebra over k. Let A be a quantum commutative H-module algebra, and V an n-dimensional left H-module so that $u = \sum S(R^2)R^1$ acts on V as the identity. Assume* $\operatorname{Char} k = 0$ *or* $\operatorname{Char} k > n$. *Let $S, T \in \operatorname{End}_{A\#H}(A \otimes V)$, then:*

1. $\det(T) \in A^H \subseteq Z(A)$.
2. $\det(T)$ *is independent of the choice of basis of* $\bigwedge_R^n(A \otimes V)$ *as a left A-module.*
3. $\det(I) = 1$.
4. $\det(S \circ T) = (\det T)(\det S)$.

When H is finite dimensional there is a complete duality between these notions.

In Section 3 we specialize to group gradings. We find in Theorem 3.2 a concrete form of this determinant for $H = kG$, G an abelian group with a symmetric bicharacter . An example of this set-up is $G = Z_n \times Z_n$ and $A = C_q[x, y]$ with $xy = q^{-1}yx$, $q^n = 1$, the well known quantum plane. We illustrate the theorem by computing $\det(T_{x+y})$, where T_{x+y} denotes right multiplication of A by $x + y$. As expected by different considerations $\det(T_{x+y}) = (x^n + y^n)^n$.

In Section 4 we use Theorem 2.19 to prove the generalization of Noether's theorem. We prove the following:

THEOREM 4.7: *Let (H, R) be a triangular n-dimensional semisimple Hopf algebra over k, where* $\operatorname{Char} k = 0$ *or* $\operatorname{Char} k > n$. *Let A be a quantum commutative H-module algebra; then:*

1. *A is integral over A^H.*
2. *A is a PI ring.*

and

THEOREM 4.8: *If (H, R) and A are as in Theorem 4.7 and A is k-affine, then:*

1. *A^H is k-affine.*
2. *A is a finitely generated left and right A^H module.*
3. *A is a left and right Noetherian PI ring.*

Some examples of the algebra A to which Theorem 4.8 applies are:

1. The R-symmetric algebra $S_R(V)$, where V is any finite-dimensional module over (H, R), and
2. The quantum plane $\mathbb{C}_q[x, y]$.

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1. Preliminaries

Let H be a Hopf algebra over a field k, with comultiplication Δ, counit ε and antipode S. We use Sweedler's [Sw] notation, leaving out parentheses in the summation notation. Denote by ${}_H\mathcal{M}$ the category of left H-modules. If $V, W \in {}_H\mathcal{M}$ then so is $V \otimes W$, where the action of H on $V \otimes W$ is defined by: $h \cdot (v \otimes w) = \sum h_1 \cdot v \otimes h_2 \cdot w$, all $h \in H$, $v \in V$, $w \in W$.

An algebra $A \in {}_H\mathcal{M}$ is an H-module algebra (or, H acts on A) if its multiplication is an H-module map; that is, if $h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b)$ all $h \in H$, $a, b \in A$. For such A, there exist two associated algebras:

1. The smash product $A\#H$ [Sw, p. 55] which is $A \otimes H$ as a vector space. Multiplication is defined by $(a\#h)(b\#g) = \sum a(h_1 \cdot b)\#h_2 g$, all $a, b \in A$, $g, h \in H$.

2. The algebra of H-invariants $A^H = \{a \in A | h \cdot a = \varepsilon(h)a, \text{ all } h \in H\}$.

Both A and H are naturally embedded in $A\#H$.

Dually, a right H-comodule is a vector space V with a structure map

$$\rho\colon V \mapsto V \otimes H, \quad \rho(v) = \sum v_0 \otimes v_1, \qquad \text{all } v \in V$$

which is coassociative; that is: $(\rho \otimes \mathrm{Id}) \circ \rho = (\mathrm{Id} \otimes \Delta) \circ \rho$. Also $(\mathrm{Id}_V \otimes \varepsilon)\rho = \mathrm{Id}_V$. A comodule map is a map $f\colon V \mapsto W$ such that $\rho_W \circ f = (f \otimes \mathrm{Id}) \circ \rho_V$.

Denote by $\mathcal{M}^H$ the category of right H-comodules. The tensor product of V and W in $\mathcal{M}^H$ is an H-comodule with structure map:

$$\rho(v \otimes w) = \sum v_0 \otimes w_0 \otimes v_1 w_1. \tag{1}$$

An algebra $A \in \mathcal{M}^H$ is an H-comodule algebra if the multiplication in A is an H-comodule map; that is: $\rho(ab) = \sum a_0 b_0 \otimes a_1 b_1$, all $a, b \in A$.

Recall that if H is finite-dimensional then H^* is a Hopf algebra as well. Denote by $\langle\ ,\ \rangle$ the evaluation of H^* on H. It is well known that if $V \in \mathcal{M}^{H^*}$ then $V \in {}_H\mathcal{M}$ by defining

(2) $$h \cdot v = \sum \langle v_1, h\rangle v_0,$$

all $h \in H$, $v \in V$.

In recent years there has been great interest in quasi-triangular Hopf algebras (quantum groups), which are neither commutative nor cocommutative Hopf algebras. The following definition is due to Drinfeld.

Definition 1.1: A **quasitriangular** Hopf algebra is a pair (H, R), where H is a Hopf algebra over k and $R = \sum R^1 \otimes R^2 \in H \otimes H$ is invertible, such that the following holds ($r = R$):

QT1. $\sum \Delta(R^1) \otimes R^2 = \sum R^1 \otimes r^1 \otimes R^2 r^2$,

QT2. $\sum R^1 \otimes \Delta(R^2) = \sum R^1 r^{(1)} \otimes r^2 \otimes R^2$,

QT3. $\Delta^{cop}(h) = R\Delta(h)R^{-1}$, all $h \in H$, where $\Delta^{cop}(h) = \sum h_2 \otimes h_1$.

QT4. If $R^{-1} = \sum R^2 \otimes R^1$ then (H, R) is called a **triangular** Hopf algebra.

It is a consequence of the above that $R^{-1} = \sum S(R^1) \otimes R^2$, and that $\sum \varepsilon(R^1)R^2 = \sum R^1 \varepsilon(R^2) = 1$. Of special interest to us is $u = \sum S(R^2)R^1$. This element is invertible and induces S^2; that is $S^2(h) = uhu^{-1}$, all $h \in H$. When (H, R) is triangular u is a group-like element [Dr2], and $u = \sum R^1 R^2$. If H is finite dimensional and $\mathrm{Char}\, k \nmid \dim H$ then [LR] have proved that $S^2 = \mathrm{Id}$ if and only if H is semisimple (and then it is also cosemisimple). Thus in this context, u is central if and only if $S^2 = \mathrm{Id}$ if and only if H is semisimple. Moreover, in this case $u^2 = 1$ by [Ra].

We are interested in the dual notion [LT] which is better suited to deal with infinite dimensional H.

Definition 1.2: A **coquasitriangular** Hopf algebra is a pair $(H, \langle\ |\ \rangle)$ where H is a Hopf algebra over k and $B = \langle\ |\ \rangle \colon H \otimes H \to k$ is a k-linear form (braiding) which is convolution invertible in $\mathrm{Hom}_k(H \otimes H, k)$ such that the following hold:

B1. $\langle h | gl \rangle = \sum \langle h_1 | g \rangle \langle h_2 | l \rangle$.

B2. $\langle hg | l \rangle = \sum \langle g | l_1 \rangle \langle h | l_2 \rangle$.

B3. $\sum \langle h_1 | g_1 \rangle g_2 h_2 = \sum h_1 g_1 \langle h_2 | g_2 \rangle$.

B4. If $\sum \langle h_1 | g_1 \rangle \langle g_2 | h_2 \rangle = \varepsilon(g)\varepsilon(h)$ then $(H, \langle\ |\ \rangle)$ is called a **cotriangular** Hopf algebra.

When H is finite dimensional then (H,R) is quasitriangular if and only if $(H^*,\langle\ |\ \rangle)$ is coquasitriangular where $\langle\ |\ \rangle$ and R are dual notions. This is easily seen since $H\otimes H$ is naturally isomorphic to $(H^*\otimes H^*)^*$. Specifically, if (H,R) is quasitriangular, define the braiding form $\langle\,|\,\rangle$:

$$\langle a|b\rangle = \sum\langle a, R^2\rangle\langle b, R^1\rangle \quad \text{all } a,b\in H^*. \tag{3}$$

A trivial example of a triangular Hopf algebra is kG, with $R = 1\otimes 1$, which is semisimple when $\operatorname{Char} k \not| |G|$. An example of a cotriangular Hopf algebra is kG where G is abelian with a symmetric bicharacter $\langle\ |\ \rangle$. This Hopf algebra is commutative and cocommutative and it arises in the context of the Lie color algebras. No example of a (co)semisimple (co)triangular Hopf algebra different from kG was known until recently when Gelaki constructed in [G] a new family of semisimple triangular Hopf algebras which are neither commutative nor cocommutative, and thus are isomorphic neither to kG nor to kG^*.

The following lemma is useful in understanding some of the later notions.

Lemma 1.3: *Let (H,R) be a finite-dimensional triangular Hopf algebra. Define $\langle\,|\,\rangle$ as in (3); then:*

1. $\langle a,u\rangle = \sum\langle a_2|a_1\rangle$, *for all $a\in H^*$.*
2. *u is central in $H \iff \sum\langle a_2|a_1\rangle a_3 = \sum\langle a_3|a_2\rangle a_1$, all $a\in H^*$.*
3. *Let $V\in\mathcal{M}^{H^*}$ and consider $V\in {}_H\mathcal{M}$ as in (2). Then $v\in V$,*

$$u\cdot v = v \iff \sum\langle v_2|v_1\rangle v_0 = v \tag{4}$$

Proof: 1. Let $a\in H^*$, then

$$\langle a,u\rangle = \sum\langle a, R^1R^2\rangle = \sum\langle a_1, R^1\rangle\langle a_2, R^2\rangle = \sum\langle a_2|a_1\rangle,$$

where the last equality follows from (3).

2. By part (1) it follows that for any $h\in H$:

$$\begin{aligned}\langle a,uh\rangle &= \sum\langle a_1,u\rangle\langle a_2,h\rangle = \sum\langle a_2|a_1\rangle\langle a_3,h\rangle \quad\text{and}\\ \langle a,hu\rangle &= \sum\langle a_1,h\rangle\langle a_3|a_2\rangle.\end{aligned}$$

Hence (2) follows.

3. Let $v\in V$, then $u\cdot v = \sum\langle v_1,u\rangle v_0 = \sum\langle v_2|v_1\rangle v_0$, where the last equality follows from part (1). ∎

In the presence of triangularity or cotriangularity, the category ${}_H\mathcal{M}$ or $\mathcal{M}^H$ respectively is very nice. We define such a category in general.

Let $\mathcal{C}$ be a symmetric monoidal category [Mac, p. 180]; that is, $\mathcal{C}$ has a tensor product on its objects satisfying certain associativity conditions (the pentagon axiom) and a twist map Ψ :

$$\Psi_{X,Y}\colon X\otimes Y\mapsto Y\otimes X, \qquad \text{all } X,Y\in\mathcal{C}.$$

This map Ψ satisfies a compatibility condition with the associativity (the hexagon axiom) and moreover: $\Psi^2=\mathrm{id}$.

Following Manin [Ma] one can now define a representation of S_m on $X^{\otimes m}$ by defining the action of $(i,i+1)$ as

$$(i,i{+}1)\cdot_\Psi(X_1\otimes\cdots\otimes X_i\otimes X_{i+1}\cdots\otimes X_n)=X_1\otimes\cdots\otimes\Psi_{X_i,X_{i+1}}(X_i\otimes X_{i+1})\otimes\cdots\otimes X_n$$

where $X_i=X$, all i. Then extend the action to $\sigma\in S_m$, by representing σ as a product of elementary transpositions. Let $\bar{x}=x_1\otimes\cdots\otimes x_m\in X^{\otimes m}$; define

$$S^m_\Psi(X)=\{\bar{x}\in X^{\otimes m}|\sigma\cdot_\Psi\bar{x}=\bar{x},\ \text{all } \sigma\in S_m\}$$

and

$$\bigwedge\nolimits^m_\Psi(X)=\{\bar{x}\in X^{\otimes m}|\sigma\cdot_\Psi\bar{x}=\operatorname{sgn}(\sigma)\bar{x},\ \text{all } \sigma\in S_m\}.$$

Let $S_\Psi(X)=\sum^\oplus_{m\geq 0}S^m_\Psi(X)$ and $\bigwedge_\Psi(X)=\sum^\oplus_{m\geq 0}\bigwedge^m_\Psi(X)$. One can check that

$$S_\Psi(X)=T(X)/\langle(\mathrm{Id}-\Psi)(x_1\otimes x_2)\rangle$$

and

$$\bigwedge\nolimits_\Psi(X)=T(X)/\langle(\mathrm{Id}+\Psi)(x_1\otimes x_2)\rangle$$

where $T(X)=$ tensor algebra over X (as in [Ma, p. 71]) and x_1, x_2 are arbitrary elements of X.

From now on we assume that the objects V in $\mathcal{C}$ are vector spaces over a field k. Recall that with appropriate characteristic assumption, $S^m_\Psi(V)$ and $\bigwedge^m_\Psi(V)$ have nice descriptions:

PROPOSITION 1.4: *Let $\mathcal{C}$, Ψ, S_Ψ and $\bigwedge_\Psi$ be as above, and assume the objects V of $\mathcal{C}$ are vector spaces over the field k of characteristic 0 or characteristic $p>m$, then*

$$S^m_\Psi(V)=t_m\cdot_\Psi V^{\otimes m} \quad\text{and}\quad \bigwedge\nolimits^m_\Psi(V)=f_m\cdot_\Psi V^{\otimes m}$$

where $t_m=\frac{1}{m!}\sum_{\sigma\in S_m}\sigma$ and $f_m=\frac{1}{m!}\sum_{\sigma\in S_m}\operatorname{sgn}(\sigma)\sigma$.

Let $V \in \mathcal{C}$; the algebra $S_\Psi(V)$ is a basic example of an algebra which is commutative in the category. Precisely,

Definition 1.5: An algebra $A \in \mathcal{C}$ with a multiplication μ is commutative in the category $\mathcal{C}$ if for $a, b \in A$

$$\mu \circ (\mathrm{Id} - \Psi)(a \otimes b) = 0.$$

When Ψ is the usual twist map, then being a commutative algebra in the category means being a usual commutative algebra.

Now, let us specialize to the case of a Hopf algebra H which is either triangular or cotriangular. It is well known that ${}_H\mathcal{M}$ is a symmetric monoidal category with the twist map induced by R:

$$\begin{aligned} \Psi_R \colon V \otimes W &\to W \otimes V \\ v \otimes w &\mapsto \sum R^2 \cdot w \otimes R^1 \cdot v, \end{aligned}$$

for any $V, W \in {}_H\mathcal{M}$, $v \in V$, and $w \in W$. Note that cocommutative Hopf algebras H are trivially triangular, with $R = 1 \otimes 1$; and then Ψ_R is the usual twist map.

We shall denote S_{Ψ_R} by S_R and $\bigwedge_{\Psi_R}$ by $\bigwedge_R$. Dually, in $\mathcal{M}^H$ the commutativity constraint is induced by the braiding $B = \langle\,|\,\rangle$ via:

$$\Psi_B(v \otimes w) = \sum \langle w_1 | v_1 \rangle w_0 \otimes v_0$$

for any $V, W \in \mathcal{M}^H$, $v \in V$ and $w \in W$.

We shall denote S_{Ψ_B} by S_B and $\bigwedge_{\Psi_B}$ by $\bigwedge_B$.

In [CW] we discussed the commutative algebras in the category ${}_H\mathcal{M}$ when (H, R) is triangular. We termed such algebras "quantum commutative". Explicitly, A is such an algebra if

1. A is an H-module algebra and
2. $ab = \sum (R^2 \cdot b)(R^1 \cdot a)$, all $a, b \in A$.

Dually, when $(H, \langle\,|\,\rangle)$ is cotriangular then $A \in \mathcal{M}^H$ is quantum commutative if:

1. A is an H-comodule algebra and
2. $ab = \sum \langle b_1 | a_1 \rangle b_0 a_0$, all $a, b \in A$.

2. Ψ-exterior algebras and a determinant construction

In this section we construct determinant functions via appropriate exterior algebras. This will be carried out in two situations which are dual when the Hopf algebra is finite-dimensional.

The main ideas are proved in the first part, the cotriangular case. We choose to elaborate on such Hopf algebras for they are better suited to deal with infinite-dimensional H. In the second part of this section we outline the analogous setup for triangular Hopf algebras. Since the proofs are dual we only briefly sketch them out.

2.1 THE COTRIANGULAR CASE. Let $(H, \langle \mid \rangle)$ be a cotriangular Hopf algebra and V a right H-comodule. Let $V^{\otimes m} = V \otimes \cdots \otimes V$, m times, where the unadorned $\otimes$ means tensoring over the field k. Let S_m denote the symmetric group. As pointed out in the preliminaries there are two natural ways to make $V^{\otimes m}$ a kS_m-module:

1. Via the usual twist map which ignores H. We call this the usual representation and denote the action by .
2. Via Ψ_B which we call the B-representation of kS_m, and denote the action by $\cdot_B$ (where $B = \langle \mid \rangle$).

The usual symmetric $S(V)$, or exterior $\bigwedge(V)$, algebras of V are constructed via (1) and are well studied. For example, if V is n-dimensional then:

$$\bigwedge(V) = \bigwedge^0(V) \oplus \cdots \oplus \bigwedge^n(V),$$

where $\dim \bigwedge^i(V) = \binom{n}{i}$.

Using this, one may construct the determinant function and prove the Cayley–Hamilton theorem.

The aim of this section is to show that under certain conditions, $S(V)$ and $\bigwedge(V)$ are isomorphic to their analogues $S_B(V)$ and $\bigwedge_B(V)$. This will then be used to construct a determinant function which will yield an analogue of the Cayley–Hamilton theorem for certain endomorphisms. Assume V is finite dimensional and let $\{v^1, \cdots, v^n\}$ be a basis for V. Let

$$(5) \qquad \rho(v^i) = \sum_s v^s \otimes \beta_{s,i} = \sum (v^i)_0 \otimes (v^i)_1 \quad \text{where } \beta_{s,i} \in H.$$

It is easy to see that

$$\Delta(\beta_{s,i}) = \sum_t \beta_{s,t} \otimes \beta_{t,i}, \quad \text{and} \quad \varepsilon(\beta_{s,i}) = \delta_{s,i}. \tag{6}$$

In this notation (4) has an explicit form for:

$$\sum \langle (v^i)_2 | (v^i)_1 \rangle (v^i)_0 = \sum_t \langle \beta_{t,i} | \beta_{s,t} \rangle v^s$$

for all i. Thus

$$v = \sum \langle v_2 | v_1 \rangle v_0 \quad \text{for all } v \in V \iff \sum_t \langle \beta_{t,i} | \beta_{s,t} \rangle = \delta_{i,s} \quad \text{for all } i. \tag{7}$$

As usual, let $\rho^2(v) = (\rho \otimes \mathrm{Id})\rho(v) = (\mathrm{Id} \otimes \Delta)\rho(v)$, and define $\rho^i(v)$ inductively. In particular we may write:

$$\rho^{m-1}(v^{i_m}) = \sum_{t_1,\cdots,t_{m-1}} v^{t_1} \otimes \beta_{t_1,t_2} \otimes \cdots \otimes \beta_{t_{m-1},i_m}.$$

Let χ and χ_B denote the characters associated with the usual and B-representations of kS_m on $V^{\otimes m}$, respectively. That is $\chi(\sigma) = \mathrm{trace}(A_\sigma)$, where A_σ is the matrix associated to the action of σ with respect to a basis of $V^{\otimes m}$. We show:

THEOREM 2.1: *Let $(H, \langle\ |\ \rangle)$ be a cotriangular Hopf algebra over k. Let V be an n-dimensional right H-comodule with $\sum \langle v_2 | v_1 \rangle v_0 = v$ for all $v \in V$; then*

1. *$\chi_B(\sigma) = \chi(\sigma)$, for all $\sigma \in S_m$, all m.*
2. *If, moreover, $\mathrm{Char}\, k = 0$ or $\mathrm{Char}\, k > m$, then*

$$(V^{\otimes m}, \cdot) \quad \text{and} \quad (V^{\otimes m}, \cdot_B)$$

are isomorphic kS_m-modules.

Proof: 1. We first prove the claim for $\sigma = (1, \ldots, m)$. Let $\{v^1, \ldots, v^n\}$ be a basis for V; then $\{v^{i_1} \otimes \cdots \otimes v^{i_m}\}$ is a basis for $V^{\otimes m}$. It is immediate that in the usual representation of kS_m such an element is invariant under $(1, \ldots, m)$ if and only if $i_1 = \cdots = i_m$. Hence the number of such elements is $\dim V$. Thus $\chi(1, \ldots, m) = \dim V$.

Next consider the B-representation.

Since $(1,\ldots,m)=(1,2)\cdots(m-2,m-1)(m-1,m)$, we have:

$$\begin{aligned}&(1,\ldots,m)\cdot(v^{i_1}\otimes v^{i_2}\otimes\cdots\otimes v^{i_m})\\&=(1,2)(2,3)\cdots(m-1,m)\cdot(v^{i_1}\otimes v^{i_2}\cdots\otimes v^{i_m})\\&=(1,2)(2,3)\cdots(m-2,m-1)\cdot\sum(v^{i_1}\otimes v^{i_2}\otimes\cdots\otimes v_0^{i_m}\otimes v_0^{i_{m-1}})\langle v_1^{i_m}|v_1^{i_{m-1}}\rangle\\&=(1,2)(2,3)\cdots(m-3,m-2)\cdot\sum(v^{i_1}\otimes v^{i_2}\cdots\otimes v_0^{i_m}\otimes v_0^{i_{m-2}}\otimes v^{i_{m-1}})\\&\quad\langle v_1^{i_m}|v_1^{i_{m-2}}\rangle\langle v_2^{i_m}|v_1^{i_{m-1}}\rangle\\&=\sum v_0^{i_m}\otimes v_0^{i_1}\otimes v_0^{i_2}\otimes\cdots\otimes v_0^{i_{m-1}}\langle v_1^{i_m}|v_1^{i_1}\rangle\langle v_2^{i_m}|v_1^{i_2}\rangle\\&\quad\cdots\langle v_{m-2}^{i_m}|v_1^{i_{m-2}}\rangle\langle v_{m-1}^{i_m}|v_1^{i_{m-1}}\rangle.\end{aligned}$$

Let $\beta_{i\,j}$ be as in (5); then the coefficient of $(v^{i_1}\otimes v^{i_2}\otimes\cdots\otimes v^{i_m})$ is given by:

$$\sum_{t_i}\langle\beta_{i_1,t_2}|\beta_{i_2,i_1}\rangle\langle\beta_{t_2,t_3}|\beta_{i_3,i_2}\rangle\cdots\langle\beta_{t_{m-2},t_{m-1}}|\beta_{i_{m-1},i_{m-2}}\rangle\langle\beta_{t_{m-1},i_m}|\beta_{i_m,i_{m-1}}\rangle.$$

Hence

$$\begin{aligned}(8)\qquad \chi_B(1,2,\cdots,m)=\sum_{i_1,\cdots,i_m}\sum_{t_i}&\langle\beta_{i_1,t_2}|\beta_{i_2,i_1}\rangle\\&\langle\beta_{t_2,t_3}|\beta_{i_3,i_2}\rangle\cdots\langle\beta_{t_{m-2},t_{m-1}}|\beta_{i_{m-1},i_{m-2}}\rangle\langle\beta_{t_{m-1},i_m}|\beta_{i_m,i_{m-1}}\rangle.\end{aligned}$$

Now, since $v=\sum\langle v_2|v_1\rangle v_0$ for all $v\in V$, we apply the equivalent form in (7) to the left-most factor in the sum. We see that $t_2=i_2$ if the product is to be non-zero. Continuing from left to right, we see that $t_j=i_j$, for all j, and thus (8) equals $\sum_{i_m=1}^{\dim V}1=\dim V$. We conclude that:

$$(9)\qquad \chi_B(1,2,\cdots,m)=\chi(1,2,\cdots,m).$$

Now let $\sigma\in S_m$; then σ can be written as a product of disjoint cycles of length $r_1,\ldots,r_t$. So σ is conjugate to

$$\tau=(1,\ldots,r_1)(r_1+1,\ldots,r_1+r_2)\cdots(r_1+\cdots+r_{t-1}+1,\cdots,m).$$

Since on conjugate elements χ and χ_B have the same value, it is enough to prove the claim for τ. For each $i=1,\ldots,t$ take $\tau_i=(1,\ldots,r_i)$, $V_i=V^{\otimes r_i}$ and χ_i, $(\chi_B)_i$ the appropriate characters on V_i. Then $\chi(\sigma)=\chi(\tau)=\prod_{i=1}^t\chi_i(\tau_i)$ and $\chi_B(\sigma)=\prod_{i=1}^t(\chi_B)_i(\tau_i)$. So by (9)

$$\chi(\sigma)=\chi_B(\sigma).$$

2. If $\operatorname{Char} k = 0$ or $\operatorname{char}(k) > m$, then two finite-dimensional left kS_m-modules are isomorphic if and only if they have the same characters. We are done by (1). ∎

As a corollary we have:

COROLLARY 2.2: *Let $(H, \langle\ |\ \rangle)$ be a cotriangular Hopf algebra over k. Let V be an n-dimensional right H-comodule such that $\sum \langle v_2 | v_1 \rangle v_0 = v$ all $v \in V$. If $\operatorname{Char} k = 0$ or $\operatorname{Char} k > n$ then*

$$\bigwedge\nolimits_B(V) = \bigwedge\nolimits_B^0 + \cdots + \bigwedge\nolimits_B^n, \quad \text{where} \quad \dim \bigwedge\nolimits_B^i = \binom{n}{i},$$

all $i \geq 0$.

Proof: By Proposition 1.4, for all i, $\bigwedge_B^i = f_i \cdot_B V^{\otimes i}$ and $\bigwedge^i = f_i \cdot V^{\otimes i}$, for all $i \leq n$. By Theorem 2.1, $f_i \cdot_B V^{\otimes i} \cong f_i \cdot V^{\otimes i}$, hence $\dim \bigwedge_B^i = \dim f_i \cdot V^{\otimes i} = \binom{n}{i}$. ∎

A related symmetric monoidal category arises as follows: Let $(H, \langle\ |\ \rangle)$ be cotriangular, and let $A \in \mathcal{M}^H$ be quantum commutative. Let $W \in \mathcal{M}^H$ so that W is a left A-module and

$$\rho(a \cdot w) = \sum a_0 \cdot w_0 \otimes a_1 w_1. \tag{10}$$

Denote the category of such W by ${}_A\mathcal{M}$. The morphisms in the category are left A-module right H-comodule maps f.

From now on, if there is no danger of ambiguity we shall repress the B in $\cdot_B$. Mimicking [CW, 2.5] we can show

LEMMA 2.3: *Let $(H, \langle\ |\ \rangle)$, A, W be as above:*

1. *Define a right action of A on W by:*

$$w \leftarrow a = \sum \langle a_1 | w_1 \rangle a_0 \cdot w_0,$$

all $w \in W$, $a \in A$. Then this action makes W into an A-A-bimodule.

2. *If $M, N \in {}_A\mathcal{M}^H$, then $M \otimes_A N \in {}_A\mathcal{M}^H$, where*

$$a \cdot (m \otimes_A n) = a \cdot m \otimes n \quad \text{and} \quad \rho(m \otimes_A n) = \sum (m_0 \otimes_A n_0) \otimes m_1 n_1,$$

for all $m \in M$, $n \in N$, $a \in A$.

3. *If $V \in \mathcal{M}^H$, then $W = A \otimes V$ is an A-A-bimodule by:*

$$a \cdot (b \otimes v) = ab \otimes v, \quad (b \otimes v) \leftarrow a = \sum \langle b_1 v_1 | a_1 \rangle (a_0 b_0 \otimes v_0)$$

all $a, b \in A$, $v \in V$.

4. *Let X be an A-A-bimodule and Y a k-vector space; then $X \otimes_A A \otimes Y$ is isomorphic as a left A-module to $X \otimes Y$ where θ: $X \otimes Y \to X \otimes_A A \otimes Y$ is given by:*

$$x \otimes y \mapsto x \otimes_A 1 \otimes y, \qquad \text{all } x \in X,\ y \in Y.$$

5. *Let $V_1, \ldots, V_m$ be right H-comodules; then*

$$\theta\colon A \otimes (V_1 \otimes \cdots \otimes V_m) \to (A \otimes V_1) \otimes_A (A \otimes V_2) \otimes_A \cdots \otimes_A (A \otimes V_m)$$

given by:

$$a \otimes (v_1 \otimes \cdots \otimes v_m) \mapsto (a \otimes v_1) \otimes_A (1 \otimes v_2) \otimes_A \cdots \otimes_A (1 \otimes v_m)$$

all $v_i \in V_i$, $a \in A$, is an isomorphism in ${}_A\mathcal{M}^H$.

Proof: (1), (2) and (3) can be proved as in [CW, 2.5], (4) is well known, (5) is easily checked by induction. ∎

We prove that ${}_A\mathcal{M}^H$ is a symmetric monoidal category.

THEOREM 2.4: *Let $(H, \langle\ |\ \rangle)$ be a cotriangular Hopf algebra, A a quantum-commutative H-comodule algebra.*

1. *For $W \in {}_A\mathcal{M}^H$ define*

$$\begin{aligned}(i, i+1) \cdot (w_1 \otimes_A \cdots \otimes_A w_i \otimes w_{i+1} \otimes_A \cdots \otimes_A w_m) \\ = w_1 \otimes_A \cdots \otimes_A \Psi_B(w_{i+1} \otimes_A w_i) \otimes_A \cdots \otimes_A w_m\end{aligned}$$

all $w_i \in W$. Let Ψ be induced by this action; then $({}_A\mathcal{M}^H, \otimes_A, \Psi)$ is a symmetric monoidal category.

2. *For each $x \in kS_m$, the action of x is a morphism in the category.*

Proof: 1. As in [CW, Th. 2.5] the category is symmetric monoidal once we show that the action of kS_m is well defined. That is, we must show that:

$$(1, 2) \cdot [(v \otimes_A (a \cdot w)] = (1, 2) \cdot [(v \leftarrow a) \otimes w)]$$

for all $v, w \in W$, $a \in A$. Well,

$$\begin{aligned}(1,2)\cdot[v\otimes_A(a\cdot w)] \\ &= \sum\langle a_1w_1|v_1\rangle(a_0\cdot w_0)\otimes v_0 \\ &= \sum\langle w_1|v_1\rangle\langle a_1|v_2\rangle(a_0\cdot w_0)\otimes v_0 && \text{(using B2).}\end{aligned}$$

On the other hand,

$$\begin{aligned}(1{,}2)\cdot[(v\leftarrow a)\otimes_A w] \\ &= (1,2)\cdot\sum\langle a_1|v_1\rangle(a_0\cdot v_0)\otimes_A w \\ &= \sum\langle w_1|a_1v_1\rangle\langle a_2|v_2\rangle w_0\otimes_A(a_0\cdot v_0) \\ &= \sum\langle w_1|a_1v_1\rangle\langle a_2|v_2\rangle(w_0\leftarrow a_0)\otimes_A v_0) \\ &= \sum\langle w_2|a_2v_1\rangle\langle a_3|v_2\rangle\langle a_1|w_1\rangle(a_0\cdot w_0)\otimes_A v_0 \\ &= \sum\langle w_2|a_2\rangle\langle w_3|v_1\rangle\langle a_3|v_2\rangle\langle a_1|w_1\rangle(a_0\cdot w_0)\otimes_A v_0 && \text{(using B1)} \\ &= \sum\langle w_1|v_1\rangle\langle a_1|v_2\rangle(a_0\cdot w_0)\otimes_A v_0 && \text{(using B4).}\end{aligned}$$

2. We show first that the action of kS_m is an A-module map.
Let $a \in A$, $v, w \in W$, then

$$a\cdot[(1,2)\cdot(v\otimes_A w)] = a\cdot\sum\langle w_1|v_1\rangle w_0\otimes_A v_0 = \sum\langle w_1|v_1\rangle(a\cdot w_0)\otimes_A v_0,$$

while

$$\begin{aligned}(1{,}2)\cdot[(a\cdot v)\otimes_A w] \\ &= \sum\langle w_1|a_1v_1\rangle w_0\otimes_A(a_0\cdot v_0) \\ &= \sum\langle w_1|a_1\rangle\langle w_2|v_1\rangle w_0\otimes_A(a_0\cdot v_0) && \text{(using B1)} \\ &= \sum\langle w_1|a_1\rangle\langle w_2|v_1\rangle(w_0\leftarrow a_0)\otimes_A v_0 \\ &= \sum\langle w_2|a_2\rangle\langle w_3|v_1\rangle\langle a_1|w_1\rangle(a_0\cdot w_0)\otimes_A v_0 \\ &= \sum\langle w_1|v_1\rangle(a\cdot w_0)\otimes_A v_0 && \text{(using B4).}\end{aligned}$$

We show next that it is an H-comodule map, that is, we show that

$$((1,2)\otimes\mathrm{Id})\rho(v\otimes w) = \rho((1,2)\cdot(v\otimes w)).$$

Well,

$$\begin{aligned}((1{,}2)\otimes\mathrm{Id})\rho(v\otimes w) \\ &= \sum((1,2)\otimes\mathrm{Id})(v_0\otimes w_0\otimes v_1w_1) \\ &= \sum\langle w_1|v_1\rangle w_0\otimes v_0\otimes v_2w_2,\end{aligned}$$

while

$$\rho((1,2)\cdot(v\otimes w)) = \rho(\sum\langle w_1|v_1\rangle w_0\otimes v_0) = \sum\langle w_2|v_2\rangle w_0\otimes v_0\otimes w_1v_1.$$

Equality follows by B3.

It remains to show that

$$\rho(1,2)\cdot[a\cdot(v\otimes w)] = \sum((1,2)\otimes \mathrm{Id})\cdot[(a\cdot(v\otimes w))_0\otimes(a\cdot(v\otimes w))_1].$$

This is easily seen following the same proof as in part *(2)*, using (1) and (10). ∎

In the following we show that the B-actions of $\sigma\in S_m$ on the k-tensor $A\otimes V^{\otimes m}$ and on the A-tensor $(A\otimes V)^{\otimes_A m}$ (as defined above) are essentially the same. Precisely:

LEMMA 2.5: *Let H, A, V, θ be as in Lemma 2.3(5), then for each $\sigma\in S_m$, the following diagram is commutative*

$$\begin{tikzcd} A\otimes V^{\otimes m} \arrow[r, "\mathrm{Id}\otimes\sigma"] \arrow[d, "\theta"'] & A\otimes V^{\otimes m} \arrow[d, "\theta"] \\ (A\otimes V)^{\otimes_A m} \arrow[r, "\sigma"] & (A\otimes V)^{\otimes_A m} \end{tikzcd}$$

where all maps are isomorphisms in the category ${}_A\mathcal{M}^H$.

Proof: We prove the lemma for $\sigma=(1,2)$; the rest follows similarly. Let $v^1,\dots,v^m\in V$, $a\in A$, then:

$$\begin{aligned}
&(1,2)\cdot\theta(a\otimes v^1\otimes\cdots\otimes v^m)\\
&\quad=(1,2)\cdot((a\otimes v^1)\otimes_A(1\otimes v^2)\otimes_A\cdots\otimes_A(1\otimes v^m))\\
&\quad=(1,2)\cdot(a[(1\otimes v^1)\otimes_A(1\otimes v^2)\otimes_A\cdots\otimes_A(1\otimes v^m)])\\
&\quad=a(1,2)\cdot[(1\otimes v^1)\otimes_A(1\otimes v^2)\otimes_A\cdots\otimes_A(1\otimes v^m)] \qquad \text{(using Theorem 2.4)}\\
&\quad=\sum a\langle v^2_1|v^1_1\rangle(1\otimes v^2_0)\otimes_A(1\otimes v^1_0)\otimes_A\cdots\otimes_A(1\otimes v^m)\\
&\quad=\theta(\mathrm{Id}\otimes(1,2))(a\otimes v_1\otimes v_2\otimes\cdots\otimes v_m).
\end{aligned}$$

Since θ, and σ are ${}_A\mathcal{M}^H$ morphisms, so are all the maps in the diagram. ∎

Since $({}_A\mathcal{M}^H,\otimes_A,\Psi)$ is a symmetric monoidal category, we can define $\bigwedge_B(W)$ for any $W\in_A\mathcal{M}^H$. As a consequence of Theorem 2.4 and Lemma 2.5 we have for $W=A\otimes V$:

COROLLARY 2.6: *Let $(H, \langle\ |\ \rangle)$ be a cotriangular Hopf algebra, A a quantum commutative H-comodule algebra and $V \in \mathcal{M}^H$; then:*

$$\bigwedge_B^i (A \otimes V) \cong A \otimes \bigwedge_B^i (V),$$

where the isomorphism is a category morphism.

The following is the key to what follows.

COROLLARY 2.7: *Let $(H, \langle\ |\ \rangle)$ be a cotriangular Hopf algebra over k. Let A be a quantum-commutative H-comodule algebra, and V an n-dimensional H-comodule so that $\sum \langle v_2 | v_1 \rangle v_0 = v$ for all $v \in V$. If $\operatorname{Char} k = 0$ or $\operatorname{Char} k > n$ then:*

$$\bigwedge_B^n (A \otimes V) = f_n \cdot (A \otimes V)^{\otimes_A^n} = A \cdot m = m \leftarrow A,$$

where m is a basis of $\bigwedge_B^n (A \otimes V)$ as both a left and right A-module and f_n is as defined in Proposition 1.4. Moreover, $\rho(m) = m \otimes h$, where h is a group-like element of H.

Proof: By Proposition 1.4, $\bigwedge_B^n (A \otimes V) = f_n \cdot (A \otimes V)^{\otimes_A^n}$. However, by Theorem 2.4, $f_n \cdot (A \otimes V)^{\otimes_A^n} \cong_\theta A \otimes f_n \cdot V^{\otimes n}$ as A-A-bimodules. Moreover, it follows from Corollary 2.2 that $\dim \bigwedge_B^n (V) = 1$. Hence there exists a nonzero $w \in f_n \cdot V^{\otimes n}$ which is a k-basis for the H-comodule $f_n \cdot V^{\otimes n} = \bigwedge_B^n (V)$. Since w generates a 1-dimensional comodule, $\rho(w) = w \otimes h$, some $h \in H$.

Obviously, $1 \otimes w$ is a basis of $A \otimes f_n \cdot V^{\otimes n}$ as a left A-module, however it is also a basis of $A \otimes f_n \cdot V^{\otimes n}$ as a right A-module. To see this, all we have to show is that for $a \in A$, $a \cdot (1 \otimes w) = (1 \otimes w) \leftarrow b$, some $b \in A$. Well,

$$\begin{aligned} & a \cdot (1 \otimes w) \\ & \quad = \sum a_0 \varepsilon(a_1) \varepsilon(w_1) \cdot (1 \otimes w_0) \\ & \quad = \sum \langle a_1 | w_1 \rangle \langle w_2 | a_2 \rangle a_0 \cdot (1 \otimes w_0) \qquad \text{(using B4)} \\ & \quad = \sum \langle w_1 | a_1 \rangle (1 \otimes w_0) \leftarrow a_0; \end{aligned}$$

but $\rho(w) = w \otimes h$, so the above equals

$$\sum \langle h | a_1 \rangle (1 \otimes w) \leftarrow a_0 = (1 \otimes w) \leftarrow \left(\sum \langle h | a_1 \rangle a_0\right)$$

as claimed.

Now let $m = \theta(1 \otimes w)$; then since θ is a morphism in the category ${}_A\mathcal{M}^H$, we are done. ∎

It is Corollary 2.7 which enables us to construct the determinant function, just as the exterior algebra does in the classical case.

LEMMA 2.8: *Let $T \in \mathrm{End}(A \otimes V)$ be a morphism in ${}_A\mathcal{M}^H$. Then the diagonal action of $T^{\otimes^n}$ on $(A \otimes V)^{\otimes_A^n}$ defined by:*

$$T^{\otimes^n}(x_1 \otimes_A \cdots \otimes_A x_n) = T(x_1) \otimes_A \cdots \otimes_A T(x_n),$$

where $x_i \in A \otimes V$, is a morphism in ${}_A\mathcal{M}^H$ which commutes with the action of kS_n.

Proof: It is straightforward to check that $T^{\otimes n}$ is a morphism in ${}_A\mathcal{M}^H$. We show that $T^{\otimes n}$ commutes with the action of kS_n. Let us check this for $\sigma = (1,2)$. The rest will follow similarly. Well, T is a comodule map; that is: $\rho T(x) = \sum T(x_0) \otimes x_1$, for all $x \in A \otimes V$. Thus for all $x_i \in A \otimes V$:

$$\begin{aligned}&(1,2) \cdot T^{\otimes n}(x_1 \otimes_A x_2 \otimes_A \cdots \otimes_A x_n)\\ &= \sum \langle (x_2)_1 | (x_1)_1 \rangle T((x_2)_0) \otimes_A T((x_1)_0) \otimes_A \cdots \otimes_A T(x_n).\end{aligned}$$

This expression equals

$$T^{\otimes n}(1,2) \cdot (x_1 \otimes_A x_2 \otimes_A \cdots \otimes_A x_n)$$

by the definition of the action of $(1,2)$. ∎

As a result we get that $A \cdot m = f_n \cdot (A \otimes V)^{\otimes_A^n}$ is $T^{\otimes n}$-stable. In particular, $T^{\otimes n}(m) \in A \cdot m$; that is $T^{\otimes n}(m) = a_T m$, where $a_T \in A$. We define:

Definition 2.9: Let $(H, \langle \ | \ \rangle)$ be a cotriangular Hopf algebra over k, let A be a quantum-commutative H-comodule algebra and let V be an n-dimensional H-comodule so that $\sum \langle v_2 | v_1 \rangle v_0 = v$ for all $v \in V$. Assume $\mathrm{Char}\, k = 0$ or $\mathrm{Char}\, k > n$. Let m be a basis of $\bigwedge_B^n(A \otimes V)$ as a left A-module, let $T \in \mathrm{End}(A \otimes V)$ be a morphism in ${}_A\mathcal{M}^H$ and let $T^{\otimes n}(m) = a_T m$, where $a_T \in A$. Define

$$\det(T) = a_T.$$

We prove the main result of this section:

THEOREM 2.10: *Let $(H, \langle \mid \rangle)$ be a cotriangular Hopf algebra over k. Let A be a quantum-commutative right H-comodule algebra, and V an n-dimensional right H-comodule so that $\sum \langle v_2 | v_1 \rangle v_0 = v$, for all $v \in V$. Assume* $\mathrm{Char}\, k = 0$ *or* $\mathrm{Char}\, k > n$. *Let $S, T \in \mathrm{End}(A \otimes V)$ be morphisms in ${}_A\mathcal{M}^H$. Then*

1. $\det(T) \in A^{\mathrm{co}H} \subset Z(A)$.
2. $\det(T)$ *is independent of the choice of basis of* $\bigwedge_B^n (A \otimes V)$ *as a left A-module.*
3. $\det(I) = 1$.
4. $\det(T \circ S) = \det(T) \det(S)$.

Proof: Note that $A^{\mathrm{co}H} \subset Z(A)$. For if $a \in A^{\mathrm{co}H}$ and $b \in A$, then

$$ab = \sum \langle b_1 | 1 \rangle b_0 a = \sum \varepsilon(b_1) b_0 a = ba.$$

Since it is easy to see that $\det(T \circ S) = \det(S) \det(T)$, (4) will follow once we show (1). Obviously $\det(\mathrm{Id}) = 1$, so let us prove (1).

Let m be a basis element of $\bigwedge_B^n (A \otimes V)$; then $\rho(m) = m \otimes h$, where h is group-like. Now,

$$\begin{aligned}
&\sum (\det T)_0 \cdot m \otimes (\det T)_1 \\
&\quad = \sum (\det T)_0 \cdot m_0 \otimes (\det T)_1 m_1 S(m_2) \\
&\quad = \sum \rho((\det T) \cdot m_0)(1 \otimes S(m_1)) \\
&\quad = (\rho((\det T) \cdot m))(1 \otimes h^{-1}) \\
&\quad = (\rho(T^{\otimes n}(m)))(1 \otimes h^{-1}) \\
&\quad = ((T^{\otimes n} \otimes \mathrm{Id}) \rho(m))(1 \otimes h^{-1}) \\
&\quad = (T^{\otimes n} \otimes \mathrm{Id})(m \otimes h)(1 \otimes h^{-1}) \\
&\quad = (\det T) \cdot m \otimes 1.
\end{aligned}$$

Since $A \cdot m$ is a free A-module, the above implies that

$$\rho(\det T) = \det T \otimes 1.$$

2. Let m' be another basis of $\bigwedge_B^n (A \otimes V)$; then $m' = a \cdot m$ where $a \in A$. Since $T^{\otimes n}$ is a left A-module map, $T^{\otimes n}(m') = a \cdot T^{\otimes n}(m) = a a_T \cdot m$. By (1), $a_T \in Z(A)$, and hence this equals $a_T \cdot m'$. That is, $T^{\otimes n}(m') = a_T \cdot m'$. ∎

2.2 THE TRIANGULAR CASE. In this subsection we briefly outline the dual situation. Let (H, R) be a triangular Hopf algebra, and let $A \in {}_H\mathcal{M}$ be a quantum-commutative H-module algebra. Let $W \in {}_H\mathcal{M}$ such that W is a left A-module and $h \cdot (a \cdot w) = \sum (h_1 \cdot a)(h_2 \cdot w)$, all $a \in A$, $h \in H$ and $w \in W$. The category of such W is known to be equivalent to the category of left $A\#H$-modules. Denote this category by ${}_{A\#H}\mathcal{M}$. As in the preliminaries, kS_m acts on $V^{\otimes m}$ via Ψ_R, we denote this action by $\cdot_R$ and let χ_R be the corresponding character. Moreover, as pointed out in (4), the previous condition $\sum \langle v_2 | v_1 \rangle v_0 = v$, for all $v \in V \in \mathcal{M}^H$, is replaced in this setting by the condition: u acts on $V \in {}_H\mathcal{M}$ as the identity, where $u = \sum S(R^2)R^1$ as defined and discussed after Definition 1.1.

We start with the dual form of Theorem 2.1:

THEOREM 2.11: *Let (H, R) be a triangular Hopf algebra over k. Let V be an n-dimensional left H-module so that u acts on V as the identity; then*

1. *$\chi_R(\sigma) = \chi(\sigma)$, for all $\sigma \in S_m$, all m.*
2. *If, moreover, $\operatorname{Char} k = 0$ or $\operatorname{Char} k > n$, then $(V^{\otimes m}, \cdot)$ and $(V^{\otimes m}, \cdot_R)$ are isomorphic as kS_m-modules, all m.*

Proof: First, as in the proof of Theorem 2.1 we show (1) for $\sigma = (1, \ldots, m)$, and $\{v_1, \ldots, v_n\}$ a basis for V.

For each $h \in H$, write: $h \cdot v_i = \sum_{j=1}^n \beta_{j\,i}(h) v_j$; then:

$$\beta_{i\,j}(gh) = \sum_k \beta_{i\,k}(g)\beta_{k\,j}(h) \quad \text{for all } g, h \in H. \tag{11}$$

Moreover, since $u \cdot v = v$, for all v,

$$\beta_{i\,j}(uh) = \beta_{i\,j}(h), \quad \text{all } h \in H \quad \text{all } 1 \le i, j \le n. \tag{12}$$

Let $R_i = \sum R_i^1 \otimes R_i^2 = R = \sum R^1 \otimes R^2$, $1 \le i \le m$. Applying (11) yields

$$\begin{aligned}
&(1, \cdots, m) \cdot_R (v_{i_1} \otimes \cdots \otimes v_{i_m}) \\
&\quad = \sum (R_{m-1}^2 \cdots R_1^2) \cdot v_{i_m} \otimes R_{m-1}^1 \cdot v_{i_1} \otimes \cdots \otimes R_1^1 \cdot v_{im-1} \\
&\quad = \sum \beta_{i_1\, i_m}(R_{m-1}^2 \cdots R_1^2)\beta_{i_2\, i_1}(R_{m-1}^1) \cdots \beta_{i_m\, i_{m-1}}(R_1^1)(v_{i_1} \otimes \cdots \otimes v_{i_m}) + y
\end{aligned}$$

where y is a linear combination of the elements other than $v_{i_1} \otimes \cdots \otimes v_{i_m}$ in the basis $\{v_j \otimes v_k \otimes \cdots \otimes v_l | 1 \leq j,k,l \leq m\}$. Thus

$$\begin{aligned}
&\chi_R(1,\ldots,m) \\
&= \sum \beta_{i_1\, i_m}(R^2_{m-1}\cdots R^2_1)\beta_{i_2\, i_1}(R^1_{m-1})\cdots\beta_{i_m\ i_{m-1}}(R^1_1) \\
&= \sum \beta_{i_2\, i_m}(R^1_{m-1}R^2_{m-1}\cdots R^2_1)\cdots\beta_{i_m\ i_{m-1}}(R^1_1) && \text{(using (11))} \\
&= \sum \beta_{i_2\, i_m}(u\cdots R^2_1)\cdots\beta_{i_m\ i_{m-1}}(R^1_1) && \text{(since } u=\textstyle\sum R^1R^2) \\
&= \sum \beta_{i_2\, i_m}(R^2_{m-2}\cdots R^2_1)\cdots\beta_{i_m\ i_{m-1}}(R^1_1) && \text{(using (12))}.
\end{aligned}$$

Continuing as above, we get

$$\chi_R(1,\ldots,m) = \sum_{i_m=1}^{\dim V} \beta_{i_m\ i_m}(u) = \dim V = \chi(1,\cdots,m). \qquad \blacksquare$$

COROLLARY 2.12: *Let (H,R) be a triangular Hopf algebra over k. Let V be an n-dimensional left H-module such that u acts on V as the identity. If* $\operatorname{Char} k = 0$ *or* $\operatorname{Char} k > n$ *then*

$$\bigwedge\nolimits_R(V) = \bigwedge\nolimits_R^0 + \cdots + \bigwedge\nolimits_R^n, \quad \text{where} \quad \dim \bigwedge\nolimits_R^i = \binom{n}{i},$$

all $i \geq 0$.

Proof: The proof follows as in Corollary 2.2, replacing B with R. $\blacksquare$

With the following definitions we have the dual of Lemma 2.3 which was proved in [CW] for the triangular case. For each $W \in {}_{A\#H}\mathcal{M}$, $a \in A$ define

$$w \leftarrow a = \sum (R^2 \cdot a)(R^1 \cdot w).$$

For $M, N \in {}_{A\#H}\mathcal{M}$, define $a\cdot(m\otimes_A n) = am\otimes_A n$ and $h\cdot(m\otimes_A n) = \sum(h_1\cdot m)\otimes_A (h_2\cdot n)$, all $m \in M$, $n \in N$, $a \in A$. If $V \in {}_H\mathcal{M}$, let $W = A\otimes V$. For $a,b \in A$, define $a\cdot(b\otimes_A v) = ab\otimes_A v$, and $(b\otimes_A v) \leftarrow a = \sum(R^2\cdot a)(R^1_1\cdot b)\otimes_A (R^1_2\cdot v)$. We have:

THEOREM 2.13: *Let (H,R) be a triangular Hopf algebra and A a quantum commutative H-module algebra.*

1. *For $W \in {}_{A\#H}\mathcal{M}$ define*

$$\begin{aligned}
&(i,i+1)\cdot(v_1\otimes_A\cdots\otimes_A v_i\otimes v_{i+1}\otimes_A\cdots\otimes_A v_m) \\
&= \sum v_1\otimes_A\cdots\otimes_A \Psi_R(v_{i+1}\otimes_A v_i)\otimes_A\cdots\otimes_A v_m
\end{aligned}$$

all $v_i \in W$. Let Ψ be induced by this action; then $({}_{A\#H}\mathcal{M}, \otimes_A, \Psi)$ is a symmetric monoidal category.

2. *The action of kS_m induced by the above is a morphism in the category.*

Proof: 1. We need to show that the action of kS_m is well defined. That is, we must show that $(1,2)\cdot(v\otimes_A (a\cdot w)) = (1,2)\cdot(v \leftarrow a)\otimes w)$, all $a \in A$ and $v, w \in W$. In $A\#H$ write ah for $a\#h$; then $ha = \sum(h_1\cdot a)h_2$. Well,

$$\begin{aligned}
&(1,2)\cdot[v\otimes_A (a\cdot w)]\\
&\quad = \sum(R^2\cdot(a\cdot w))\otimes_A (R^1\cdot v)\\
&\quad = \sum((R^2_1\cdot a)\cdot(R^2_2\cdot w))\otimes_A (R^1\cdot v)\\
&\quad = \sum((r^2\cdot a)\cdot(R^2\cdot w))\otimes_A (R^1 r^1\cdot v) \qquad \text{(using QT2)},
\end{aligned}$$

while

$$\begin{aligned}
&(1,2)\cdot[(v\leftarrow a)\otimes_A w]\\
&\quad = \sum(R^2\cdot w)\otimes_A (R^1\cdot(v\leftarrow a))\\
&\quad = \sum(R^2\cdot w)\otimes_A (R^1\cdot[(r^2\cdot a)\cdot(r^1\cdot v)]\\
&\quad = \sum(R^2\cdot w)\otimes_A (R^1_1 r^2\cdot a)\cdot(R^1_2 r^1\cdot v)\\
&\quad = \sum(R^2T^2\cdot w)\otimes_A [(R^1 r^2\cdot a)\cdot(T^1 r^1\cdot v)] \qquad \text{(using QT1)}\\
&\quad = \sum[(R^2T^2\cdot w)\leftarrow(R^1 r^2\cdot a)]\otimes_A (T^1 r^1\cdot v)\\
&\quad = \sum[(S^2R^1r^2\cdot a)\cdot(S^1R^2T^2\cdot w)]\otimes_A (T^1 r^1\cdot v)\\
&\quad = \sum(r^2\cdot a)\cdot(T^2\cdot w)\otimes_A (T^1 r^1\cdot v) \qquad \text{(using QT4)}.
\end{aligned}$$

2. We show first that the action of kS_m is an A-module map. Let $a \in A$ and $v, w \in W$; then

$$a\cdot[(1,2)\cdot(v\otimes_A w)] = \sum(aR^2\cdot w)\otimes_A (R^1\cdot v),$$

while

$$
\begin{aligned}
&(1,2)\cdot[a\cdot(v\otimes_A w)]\\
&\quad=\sum R^2\cdot w\otimes_A R^1\cdot(a\cdot v)\\
&\quad=\sum R^2\cdot w\otimes_A (R^1_1\cdot a)(R^1_2\cdot v)\\
&\quad=\sum R^2r^2\cdot w\otimes_A (R^1\cdot a)(r^1\cdot v) && \text{(using QT1)}\\
&\quad=\sum (R^2r^2\cdot w\leftarrow (R^1\cdot a))\otimes_A (r^1\cdot v)\\
&\quad=\sum (T^2R^1\cdot a)\cdot(T^1R^2r^2\cdot w)\otimes_A (r^1\cdot v)\\
&\quad=\sum (ar^2\cdot w)\otimes_A (r^1\cdot v) && \text{(using QT4).}
\end{aligned}
$$

The proof of the fact that the action of kS_m is an H-module map follows from QT3. ∎

LEMMA 2.14: *Let (H,R) be a triangular Hopf algebra over k and let $V\in {}_H\mathcal{M}$. Let A be a quantum-commutative H-module algebra and let θ be as in Lemma 2.3(5); then for each $\sigma\in S_m$, the following diagram is commutative:*

$$
\begin{tikzcd}
A\otimes V^{\otimes m} \arrow[r, "\mathrm{Id}\otimes\sigma"] \arrow[d, "\theta"'] & A\otimes V^{\otimes m} \arrow[d, "\theta"] \\
(A\otimes V)^{\otimes_A m} \arrow[r, "\sigma"] & (A\otimes V)^{\otimes_A m}
\end{tikzcd}
$$

where all maps are isomorphisms in the category ${}_{A\#H}\mathcal{M}$.

Proof: The proof follows from Theorem 2.13 in the same way as Lemma 2.5 follows from Theorem 2.4 ∎

COROLLARY 2.15: *Let $(H,R),A,V$ be as in Lemma 2.14; then $\bigwedge^i_R(A\otimes V)\cong A\otimes\bigwedge^i_R(V)$ where the isomorphism is a category morphism.*

COROLLARY 2.16: *Let $(H,R),A,V$ be as above. If $\operatorname{Char}k=0$ or $\operatorname{Char}k>n$, then:*

$$\bigwedge\nolimits^n_R(A\otimes V)=f_n\cdot(A\otimes V)^{\otimes^n_A}=A\cdot m=m\leftarrow A$$

where m is a basis of $\bigwedge^n_R(A\otimes V)$ as both a left and right A-module and f_n is as defined in Proposition 1.4. Moreover, there exists $\lambda\in G(H^*)$ so that $h\cdot m=\langle\lambda,h\rangle m$, for all $h\in H$.

Proof: As in the proof of Corollary 2.7. By Proposition 1.4, Theorem 2.13 and the dual form of Corollary 2.2, $\dim \bigwedge_R^n(V) = 1$. Hence there exists $w \in f_n \cdot V^{\otimes n}$ which is a k-basis for the H-module $f_n \cdot V^{\otimes n} = \bigwedge_R^n(V)$. Thus $h \cdot w \in kw$ for each $h \in H$. That is, $h \cdot w = \langle \lambda, h\rangle w$ where $\lambda \in H^*$. We need to show that $1 \otimes w$ is a right A-basis of $A \otimes f_n \cdot V^{\otimes n}$ as well. That is, we must show that for any $a \in A$, $a \cdot (1 \otimes w) = (1 \otimes w) \leftarrow b$, some $b \in A$. Well,

$$\begin{aligned}
&a\cdot(1 \otimes w)\\
&= \sum (R^2 r^1 \cdot a) \cdot (1 \otimes R^1 r^2 \cdot w) \qquad\qquad \text{(by QT4)}\\
&= \sum (1 \otimes r^2 \cdot w) \leftarrow (r^1 \cdot a)\\
&= \sum (1 \otimes \langle \lambda, r^2\rangle w) \leftarrow (r^1 \cdot a)\\
&= (1 \otimes w) \leftarrow \sum \langle \lambda, r^2\rangle r^1 \cdot a
\end{aligned}$$

as claimed. ∎

COROLLARY 2.17: *Let $T \in \mathrm{End}_{A\#H}(A \otimes V)$, and let $T^{\otimes n}$ act diagonally on $(A \otimes V)_A^{\otimes n}$. Then $T^{\otimes n}$ is an $A\#H$-module map that commutes with the action of kS_m.*

Proof: It is straightforward that $T^{\otimes n}$ is an $A\#H$-module map; it commutes with the action of kS_m since this action is defined via R (i.e. via the H-action). ∎

Let $m = \theta(1 \otimes w)$. By the above remark, $A \cdot m = f_n \cdot (A \otimes V)^{\otimes_A^n}$ is stable under $T^{\otimes n}$. In particular $T^{\otimes n}(m) = a_T m$, where $a_T \in A$.

Definition 2.18: Let (H, R) be a triangular Hopf algebra over k, let A be a quantum-commutative H-module algebra and let V be an n-dimensional H-module so that u acts on V as the identity. Assume $\mathrm{Char}\, k = 0$ or $\mathrm{Char}\, k > n$. Let m be a basis of $\bigwedge_R^n(A \otimes V)$ as a left A-module, let $T \in \mathrm{End}(A \otimes V)$ be a morphism in ${}_A\mathcal{M}^H$ and let $T^{\otimes n}(m) = a_T m$, where $a_T \in A$. Define

$$\det(T) = a_T.$$

THEOREM 2.19: *Let (H, R) be a triangular Hopf algebra over k. Let A be a quantum-commutative H-module algebra, and V an n-dimensional left H-module so that u acts on V as the identity. Assume* $\mathrm{Char}\, k = 0$ *or* $\mathrm{Char}\, k > n$. *Let $S, T \in \mathrm{End}_{A\#H}(A \otimes V)$, then:*

1. $\det(T) \in A^H \subseteq Z(A)$.
2. $\det(T)$ *is independent of the choice of basis of* $\bigwedge_R^n(A \otimes V)$ *as a left* A*-module.*
3. $\det(I) = 1$.
4. $\det(S \circ T) = (\det T)(\det S)$.

Proof: All we have to show is (1). The rest is the same as in the proof of Theorem 2.10. Let $h \in H$; then

$$\begin{aligned}
&(h\cdot \det(T)) \cdot m \\
&\quad = \sum h_1 \cdot (\det(T) s(h_2) \cdot m) \\
&\quad = \sum h_1 \cdot (\det(T) \langle \lambda, s(h_2)\rangle m) && \text{(using Corollary 2.16)} \\
&\quad = \sum \langle \lambda, s(h_2)\rangle h_1 \cdot (T^{\otimes n}(m)) \\
&\quad = \sum \langle \lambda, s(h_2)\rangle T^{\otimes n}(h_1 \cdot m) \\
&\quad = \sum \langle \lambda, s(h_2)\rangle \langle \lambda, h_1\rangle T^{\otimes n}(m) \\
&\quad = \varepsilon(h) \det(T) m && \text{(since } \lambda \in G(H^*)).
\end{aligned}$$

Hence $h \cdot \det(T) = \varepsilon(h)\det(T)$ as claimed. ∎

3. Applications to group gradings

In what follows we give an explicit expression of the determinant for $H = kG$, G an abelian group, possibly infinite, with a symmetric bicharacter $\langle\ |\ \rangle$. Let V be an H-comodule; then as is well known, V is G-graded. That is: $V = \sum_{g\in G}^{\oplus} V_g$ where $V_g = \{v \in V | \rho(v) = v \otimes g\}$. In particular $V_1 = V^{\operatorname{co} H}$.

The condition: $\sum \langle v_1 | v_2\rangle v_0 = v$ on V just means here that:

$$\langle g | g\rangle = 1 \tag{13}$$

for all g in the support of V (i.e $V_g \neq 0$). This assumption can be stated in the language of Lie color algebras [S] as follows: Let $G_+ = \{g \in G | \langle g|g\rangle = 1\}$ and $V_+ = \sum_{g\in G_+} V_g$, then our assumption is that $V = V_+$.

In what follows V will be finite dimensional, so we choose a basis $B = \{u_i\}$ of V which consists of homogeneous elements. That is, $\rho(u_i) = u_i \otimes g_i$, some $g_i \in G$. Let A be a quantum-commutative H-comodule algebra; that is, for $a \in A_g$, $b \in A_h$,

$$ab = \langle h|g\rangle ba. \tag{14}$$

Form $A\otimes V$, which, by (1), is also G-graded: $(A\otimes V)_g=\sum_{h\in G}A_{gh^{-1}}\otimes V_h$. Let $T\in \mathrm{End}(A\otimes V)$ be a morphism in ${}_A\mathcal{M}^H$; then for each u_i in the homogeneous basis of V:

$$T(1\otimes u_i)=\sum_{j=1}^{n}a_{ij}\otimes u_j. \tag{15}$$

Moreover, since T is a comodule map, $T(1\otimes u_i)\in (A\otimes V)_{g_i}$, and thus the above $a_{ij}\in A_{g_ig_j^{-1}}$.

Also note that for $x\in (A\otimes V)_g$ and $y\in (A\otimes V)_h$,

$$\Psi_B(x\otimes y)=\langle h|g\rangle(y\otimes x). \tag{16}$$

Hence as in the usual situation, if $\tau\in S_n$ and $x_i\in (A\otimes V)_{g_i}$,

$$\tau\cdot(x_1\otimes\cdots\otimes x_n)=\mu_\tau(x_{\tau^{-1}(1)}\otimes\cdots\otimes x_{\tau^{-1}(n)}) \tag{17}$$

where $\mu_\tau\in k$.

For simplicity of notation, let us write throughout av for $a\otimes v\in A\otimes V$. All tensors are now over A. As in Lemma 2.3, for $v\in V_g$, $a\in A_h$ and $w\in V$:

$$v\otimes aw=(v\leftarrow a)\otimes w=\langle h|g\rangle av\otimes w. \tag{18}$$

Recall that

$$\langle g|h\rangle\langle g|t\rangle=\langle g|ht\rangle. \tag{19}$$

Hence, for all $g\in G_+$, $h,t\in G$:

$$\langle gt|gh\rangle=\langle g|h\rangle\langle t|gh\rangle. \tag{20}$$

Let $\sigma=(i_1,\ldots i_k)$ be a cycle in S_n. Define

$$a_\sigma=a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ki_1}.$$

Since σ can be represented as $(i_2,\ldots,i_k,i_1)=\cdots=(i_k,i_1,\ldots,i_{k-1})$, we show first that a_σ is well defined. Consider the representation $(i_j,\ldots,i_k,i_1,\ldots,i_{j-1})$. Let $x=a_{i_j,i_{j+1}}\cdots a_{i_k,i_1}$ and $y=a_{i_1,i_2}\cdots a_{i_{j-1},i_j}$; then we claim that $xy=yx=a_\sigma$. Indeed, let $g=g_{i_j}g_{i_1}^{-1}$; then $x\in A_g$ and $y\in A_{g^{-1}}$. By (14) we have $xy=\langle g^{-1}|g\rangle yx$; but $\langle g^{-1}|g\rangle=1$ since $\langle g|g\rangle=1$. So $xy=yx$ as claimed. Now, since $a_{ij}\in A_{g_ig_j^{-1}}$, it is immediate that $a_\sigma\in A_1=A^{\mathrm{co}\,H}\subseteq Z(A)$.

If $\sigma \in S_n$, then express σ as a product of disjoint cycles $\sigma = \sigma_1 \cdots \sigma_r$ and now define

$$a_\sigma = a_{\sigma_1} \cdots a_{\sigma_r}.$$

Before we state the main theorem of this section we need a technical lemma.

LEMMA 3.1: *Let k be a field of characteristic 0 and let G be an abelian group with a symmetric bicharacter $\langle\ |\ \rangle$. Let V be n-dimensional and G-graded, so that for all $g \in$ support V, $\langle g \mid g\rangle = 1$. Let A be a G-graded algebra so that $ab = \langle h \mid g\rangle ba$ for all $a \in A_g$, $b \in A_h$. Let $B = \{u_i\}$ be a homogeneous basis of V; then*

$$\sigma \cdot (a_{1\sigma(1)}u_{\sigma(1)} \otimes \cdots \otimes a_{n\sigma(n)}u_{\sigma(n)}) = a_\sigma u_1 \otimes \cdots \otimes u_n, \tag{21}$$

all $\sigma \in S_n$.

Proof: At first we prove that if $\sigma = (i_1, \ldots, i_k)$ and $u_{i_j} \in V_{g_{i_j}}$, then

$$\begin{aligned}(a_{i_1 i_2}u_{i_2} \otimes a_{i_2 i_3}u_{i_3} \otimes \cdots \otimes a_{i_k i_1}u_{i_1}) \\ = \langle g_{i_1}^{-1} | g_{i_k} \cdots g_{i_2}\rangle a_\sigma (u_{i_2} \otimes u_{i_3} \otimes \cdots \otimes u_{i_1}).\end{aligned} \tag{22}$$

Indeed, by (18) and (19):

$$\begin{aligned}&(a_{i_1 i_2}u_{i_2} \otimes \cdots \otimes a_{i_k i_1}u_{i_1}) \\ &\quad = \langle g_{i_2} g_{i_3}^{-1} | g_{i_2}\rangle \langle g_{i_3} g_{i_4}^{-1} | g_{i_3} g_{i_2}\rangle \cdots \langle g_{i_{k-1}} g_{i_k}^{-1} | g_{i_{k-1}} g_{i_{k-2}} \cdots g_{i_2}\rangle \\ &\quad \langle g_{i_k} g_{i_1}^{-1} | g_{i_k} g_{i_{k-1}} \cdots g_{i_2}\rangle a_\sigma (u_{i_2} \otimes \cdots \otimes u_{i_k} \otimes u_{i_1}) \\ &\quad = \langle g_{i_3}^{-1} | g_{i_2}\rangle \langle g_{i_3} | g_{i_2}\rangle \langle g_{i_4}^{-1} | g_{i_3} g_{i_2}\rangle \cdots \langle g_{i_k}^{-1} | g_{i_{k-1}} \cdots g_{i_2}\rangle \\ &\quad \langle g_{i_k} | g_{i_{k-1}} \cdots g_{i_2}\rangle \langle g_{i_1}^{-1} | g_{i_k} \cdots g_{i_2}\rangle a_\sigma (u_{i_2} \otimes \cdots \otimes u_{i_1}) \qquad \text{(using (20))}.\end{aligned}$$

Since $\langle g|h\rangle\langle g^{-1}|h\rangle = 1$ for any $g, h \in G$, this equals

$$\langle g_{i_1}^{-1} | g_{i_k} \cdots g_{i_2}\rangle a_\sigma (u_{i_2} \otimes \cdots \otimes u_{i_1}).$$

Next we prove that for all $l > 0$, $u_{i_j} \in V_{g_{i_j}}$ and $x_i \in V_{g_i}$:

$$\begin{aligned}&a_\sigma (x_1 \otimes \cdots \otimes x_l \otimes u_{i_1} \otimes u_{i_2} \otimes \cdots \otimes u_{i_k} \otimes x_{l+k+1} \otimes \cdots \otimes x_n) \\ &= (l+1, \ldots, l+k) \cdot (x_1 \otimes \cdots \otimes x_l) \otimes (a_{i_1 i_2}u_{i_2} \otimes \cdots \otimes a_{i_k i_1}u_{i_1}) \\ &\qquad \otimes (x_{l+k+1} \otimes \cdots \otimes x_n).\end{aligned} \tag{23}$$

Well, since $(l+1,\dots,l+k) = (l+1,l+2)\cdots(l+k-1,l+k)$ then, by (22), the right hand side equals

$$\begin{aligned}&a_\sigma\langle g_{i_1}^{-1}|g_{i_k}\dots g_{i_2}\rangle(l+1,l+2)\cdots(l+k-1,l+k)\\&\quad\cdot(x_1\otimes\cdots\otimes x_l\otimes u_{i_2}\otimes\cdots\otimes u_{i_k}\otimes u_{i_1}\otimes x_{l+k+1}\otimes\cdots\otimes x_n)\\&=a_\sigma\langle g_{i_1}^{-1}|g_{i_k}\dots g_{i_2}\rangle(l+1,l+2)\cdots(l+k-2,l+k-1)\langle g_{i_1}|g_{i_k}\rangle\\&\quad(x_1\otimes\cdots\otimes x_l\otimes u_{i_2}\otimes u_{i_3}\cdots\otimes u_{i_1}\otimes u_{i_k}\otimes x_{l+k+1}\otimes\cdots\otimes x_n).\end{aligned}$$

Continuing similarly, this equals

$$\begin{aligned}&a_\sigma\langle g_{i_1}^{-1}|g_{i_k}\dots g_{i_2}\rangle\langle g_{i_1}|g_{i_k}\dots g_{i_2}\rangle\\&\quad(x_1\otimes\cdots\otimes x_l\otimes u_{i_1}\otimes\cdots\otimes u_{i_k}\otimes x_{l+k+1}\otimes\cdots\otimes x_n).\end{aligned}$$

Again, since $\langle g\,|\,h\rangle\langle g^{-1}\,|\,h\rangle = 1$, (23) is proved.

Let $\sigma = \sigma_1\cdots\sigma_r$ be a product of disjoint cycles where $\sigma_j = (i_{k_{j-1}+1},\dots,i_{k_j})$, $k_0 = 0$ and $1 < k_1 < \cdots < k_{r-1} < n$. Now, (23) enables us to make computations for a "normalized" form of σ:

$$\hat\sigma = (1,2,\dots,k_1)(k_1+1,\dots,k_2)\dots(k_{r-1}+1,\dots,n).$$

For, setting $\tau = \binom{1\,2\dots n}{i_1 i_2\dots i_n}$ we have $\tau\hat\sigma\tau^{-1} = \sigma$. Now, replacing σ by $\sigma\tau\tau^{-1}$ in (21) it will be necessary as a first step to replace $(a_{1\sigma(1)}u_{\sigma(1)}\otimes\cdots a_{n\sigma(n)}u_{\sigma(n)})$ by $\tau^{-1}\cdot(a_{1\sigma(1)}u_{\sigma(1)}\otimes\cdots a_{n\sigma(n)}u_{\sigma(n)})$. Note that $a_{i\sigma(i)}u_{\sigma(i)} = x_i \in (A\otimes V)_{g_i}$ and that $x_{\tau(i)} = a_{\tau(i)\sigma\tau(i)}u_{\sigma\tau(i)}$, thus by (17) the left hand side of (21) equals

$$\sigma\tau\mu_{\tau^{-1}}\cdot(a_{\tau(1)\sigma\tau(1)}u_{\sigma\tau(1)}\otimes\cdots\otimes a_{\tau(n)\sigma\tau(n)}u_{\sigma\tau(n)}).$$

By the definitions of τ and σ and the fact that $a_{\sigma_i}\in Z(A)$ this equals

$$\begin{aligned}&\sigma\tau\mu_{\tau^{-1}}\cdot(a_{i_1i_2}u_{i_2}\otimes a_{i_2i_3}u_{i_3}\otimes\cdots\otimes a_{i_ki_1}u_{i_1})\\&\quad\otimes\cdots\otimes(a_{i_{k_{r-1}+1}i_{k_{r-1}+2}}u_{i_{k_{r-1}+2}}\otimes\cdots\otimes a_{i_ni_{k_{r-1}+1}}u_{i_{k_{r-1}+1}})\\&\quad=\sigma\tau a_{\sigma_1}a_{\sigma_2}\cdots a_{\sigma_r}\hat\sigma_r^{-1}\cdots\hat\sigma_2^{-1}\hat\sigma_1^{-1}\mu_{\tau^{-1}}\cdot(u_{i_1}\otimes\cdots\otimes u_{i_n}) &&\text{(using (23))}\\&\quad=a_{\sigma_1}a_{\sigma_2}\cdots a_{\sigma_r}\sigma\tau\hat\sigma^{-1}\tau^{-1}\cdot(u_1\otimes\cdots\otimes u_n) &&\text{(using (17))}\\&\quad=a_\sigma\cdot(u_1\otimes\cdots\otimes u_n).\end{aligned}$$

This concludes the proof of the lemma. ∎

THEOREM 3.2: *Let G be an abelian group with a symmetric bicharacter $\langle\ |\ \rangle$, and k a field of characteristic* 0. *Let V be n-dimensional and G-graded, so that for all $g \in$ support V, $\langle g \,|\, g\rangle = 1$.*

Let A be a G-graded algebra so that $ab = \langle h \,|\, g\rangle ba$ for all $a \in A_g$, $b \in A_h$. Let $T \in \mathrm{End}(A \otimes V)$ be a morphism in ${}_AM^{kG}$; then

$$\det\, T = \sum_{\sigma \in S_n} \mathrm{sgn}\,(\sigma) a_\sigma.$$

Proof: By the definition of $\det(T)$ we must find a nonzero element w of $V^{\otimes n}$ so that $f_n \cdot w \neq 0$, where $f_n = \sum_{\sigma \in S_n} \mathrm{sgn}\,(\sigma)\sigma$, and then compute $f_n \cdot T^{\otimes n}(w)$. First we show that as in the case of ordinary determinants, if $x_i = x_j$ then $f_n \cdot (x_1 \otimes \cdots \otimes x_n) = 0$. Without loss of generality we may assume $x_1 = x_2$. Then

$$f_n(1,2) \cdot (x_1 \otimes \cdots \otimes x_n) = f_n \langle g_1 | g_1\rangle \cdot (x_1 \otimes \cdots \otimes x_n) = f_n \cdot (x_1 \otimes \cdots \otimes x_n).$$

On the other hand,

$$f_n(1,2) \cdot (x_1 \otimes \cdots \otimes x_n) = -f_n \cdot (x_1 \otimes \cdots \otimes x_n).$$

Thus, $f_n \cdot (x_1 \otimes \cdots \otimes x_n) = 0$. Hence, without loss of generality $w = u_1 \otimes \cdots \otimes u_n$, where $\{u_i\}$ is a homogeneous k-basis for V, is an appropriate element.

Now, since $T(u_i) = \sum a_{ij} u_j$, we have:

$$f_n \cdot T^{\otimes n}(u_1 \otimes \cdots \otimes u_n) = \sum f_n \cdot (a_{i_1 j_1} u_{j_1} \otimes a_{i_2 j_2} u_{j_2} \otimes \cdots \otimes a_{i_n j_n} u_{j_n}).$$

By the above, the only nonzero summands have all u_{j_k} distinct; hence the above sum equals

$$\begin{aligned} &\sum_{\sigma \in S_n} f_n \cdot (a_{1\sigma(1)} u_{\sigma(1)} \otimes a_{2\sigma(2)} u_{\sigma(2)} \otimes \cdots \otimes a_{1\sigma(n)} u_{\sigma(n)}) \\ &\quad = \sum_{\sigma \in S_n} f_n \cdot (\sigma^{-1} a_\sigma \cdot (u_1 \otimes \cdots \otimes u_n)) \qquad \text{(using (21))} \\ &\quad = \sum a_\sigma \mathrm{sgn}\,(\sigma) f_n \cdot (u_1 \otimes \cdots \otimes u_n). \end{aligned}$$

The last equality follows, since by Theorem 2.4(2), f_n is a left A-module map. On the other hand, by definition,

$$f_n \cdot T^{\otimes n}(u_1 \otimes \cdots \otimes u_n) = \det(T) f_n \cdot (u_1 \otimes \cdots \otimes u_n).$$

By Corollary 2.7 this implies that $\det(T) = \sum a_\sigma \mathrm{sgn}\,(\sigma)$, as claimed. ∎

Let us compute the determinant in an explicit example.

Example 3.3: Let $A = \mathbb{C}_q[x, y]$ be the quantum plane. That is, A equals the free algebra $\mathbb{C}\langle x, y\rangle$ modulo the relation $xy = q^{-1}yx$, where q is an nth root of 1. Let $G = Z_n \times Z_n = \{(i,j) | 0 \le i,j < n\}$ with the symmetric bicharacter given on generators by:

$$\langle (1,0)|(1,0)\rangle = \langle (0,1)|(0,1)\rangle = 1,\ \langle (1,0)|(0,1)\rangle = q \text{ and } \langle (0,1)|(1,0)\rangle = q^{-1}.$$

Note that $\langle g|g\rangle = 1$ for all $g \in G$.

Grade A by G by giving x "degree" $(1,0)$ and y "degree" $(0,1)$. In this grading A is quantum commutative [CW]. Explicitly: $A = \sum_{(i,j)\in G} A_1 x^i y^j$, where $A_1 = \mathbb{C}[x^n, y^n]$.

Recall that A being G-graded, G finite, is equivalent to A being a $(\mathbb{C}G)^*$-module algebra. Let $V = (\mathbb{C}G)^*$, and let $\{p_g\}$ be a basis for V, dual to the basis $\{g\}$ of $\mathbb{C}G$. Then V has a natural G-grading: $V = \sum_{g\in G} kp_g$. Since A is G-graded it follows that $A \otimes V = A \otimes (\mathbb{C}G)^*$ can be made into an algebra, the algebra $A\#(\mathbb{C}G)^*$. The multiplication in $A\#(\mathbb{C}G)^*$ is defined explicitly by:

$$(a\#p_g)(b\#p_h) = ab_{gh^{-1}}\#p_h,$$

all $a, b \in A,\ g, h \in G$. Right multiplications in $A\#(\mathbb{C}G)^*$ by elements of A are elements in ${}_A\mathcal{M}^{\mathbb{C}G}$.

So let T_a denote right multiplication by $a \in A$. We claim that

$$\det T_{x+y} = (x^n + y^n)^n.$$

Let $B = \{p_{(i,j)}\}$ be the basis $\{u_i\}$ of $V = (\mathbb{C}G)^*$ used in proof of Theorem 3.2. It is a G-homogeneous basis for V, with $p_{(i,j)} \in V_{(i,j)}$ and $\Delta(p_{(i,j)}) = \sum_{k,l} p_{(i-k,j-l)} \otimes p_{(k,l)}$. Since $x \in A_{(1,0)}$ and $y \in A_{(0,1)}$ we have:

$$T_{x+y}(p_{(i,j)}) = p_{(i,j)}(x+y) = xp_{(i-1,j)} + yp_{(i,j-1)}.$$

Thus in the notation of Theorem 3.2 we have:

$$a_{(i,j),(i-i,j)} = x \quad \text{and} \quad a_{(i,j),(i,j-1)} = y \quad \text{while } a_{(i,j),(k,l)} = 0$$

for all other k, l. Let S_{n^2} = permutation group on $\{(i,j) \,|\, 0 \le i,j \le n\}$; then for $\sigma \in S_{n^2}$ given by: $(i,j) \to (i-1,j)$ we get $a_\sigma = x^{n^2}$, as all the $a_{(i,j),\sigma(i,j)}$ equal x. For all other $\sigma \in S_{n^2}$, either $a_\sigma = 0$ or a_σ is a polynomial in x and y with total degree of x less than n^2. That is:

$$\det T_{x+y} = x^{n^2} + yP(x,y) \text{ and } \deg_x P(x,y) \le n^2 - 1. \tag{24}$$

On the other hand, since $xy = q^{-1}yx$ and $q^n = 1$, we have $(x+y)^n = x^n + y^n$ and so $(\det T_{x+y})^n = \det T_{(x+y)^n} = \det T_{x^n+y^n}$. But $x^n + y^n$ is a central element in $A\#(\mathbb{C}G)^*$, hence $\det T_{(x^n+y^n)} = (x^n+y^n)^{n^2}$. We have:

$$(\det T_{x+y})^n = (x^n + y^n)^{n^2}. \tag{25}$$

Since $\det T_{x+y} \in A^H = \mathbb{C}[x^n, y^n]$, a commutative domain, and since the ideal generated in it by $x^n + y^n$ is a prime ideal, (25) implies that $\det T_{x+y} = \alpha(x^n + y^n)^n$. Comparing this to (24) we deduce that $\alpha = 1$, and we are done.

As pointed out by the referee, this determinant is the one to be expected using standard results. For, if F is the field of fractions of $\mathbb{C}[x^n, y^n]$ and $R = AF$ is the ring of fractions of A, then R is a central simple algebra of dimension n^2 over F. The determinant obtained above is actually the determinant obtained by considering $R \subset \mathrm{End}_F(R)$ via right multiplication of R on itself. This is usually called the "reduced norm" [Ro2, pp. 174–175].

4. Integrality of A/A^H

In this section we apply results from previous sections to questions about integrality of A over A^H. We start by recalling some definitions and known results.

Definition 4.1: Integrality:

1. Let $R \subset S$, where R is central in S; then S is integral over R if each $x \in S$ satisfies a monic polynomial over R.
2. Definition (1) was generalized by Schelter for non-commutative rings $R \subset S$:

   S is Schelter integral over R if every $x \in S$ satisfies an equation of the form
   $$x^n + \sum p_i(x) = 0,$$
   where the $p_i(x)$ are R monomials in x of degree $\deg_x p_i < n$. That is, in each p_i the coefficients, which are elements of R, are allowed to be interspersed among the occurring x's.

THEOREM 4.2 ([BV]): *Let $R \subseteq S \subseteq B$ be PI rings such that B is Schelter integral over S and S is Schelter integral over R. Then B is Schelter integral over R.*

It is well known that if a finite group, G, acts on a commutative algebra A, then A is integral over A^{G}. Noether's theorem moreover assures that if A is k-affine then so is A^G. These were generalized to:

THEOREM 4.3 ([DG, F-S]): *Let H be a finite-dimensional, cocommutative Hopf algebra and let A be a commutative H-module algebra. Then*

1. *A/A^H is integral.*
2. *If A is k-affine then A^H is k-affine.*

In the sequel we also use the following form of the **Artin–Tate Lemma**:

THEOREM 4.4 ([MR, p. 481]): *Let $A \subset B \subset C$ be rings such that*

1. *A, B are central subrings of C.*
2. *C is an affine A-algebra.*
3. *C is a finitely generated B-module.*
4. *A is Noetherian.*

Then B is an affine A-algebra.

In what follows we shall generalize Theorem 4.3, replacing cocommutativity of H by triangularity and commutativity of A by quantum commutativity. We assume that H is finite-dimensional, hence triangularity or cotriangularity are completely dual notions. We choose throughout this section to discuss triangular (H, R). We apply results of previous sections to the H-modules $V = H$ (acting on itself by left multiplication) or to $V = A$, in order to prove integrality of A over A^H and related properties. First, we wish to find a convenient basis for $\bigwedge_R^n(H)$. We start with some basic lemmas.

LEMMA 4.5: *Let (H, R) be a quasitriangular Hopf algebra; then R can be written as follows:*

$$R = \sum_{\alpha \in I} h_\alpha \otimes g_\alpha + 1 \otimes 1$$

where $\{h_\alpha\}$ and $\{g_\alpha\}$ are linearly independent subsets of $\operatorname{Ker} \varepsilon$

Proof: Since $K = \operatorname{Ker} \varepsilon$ has codimension 1, we may choose a basis for H of the form

$$\{1, h_\alpha | h_\alpha \in K, \alpha \in \text{some index set } I\}.$$

Write $R = 1 \otimes h + \sum_\alpha h_\alpha \otimes g_\alpha$, some $g_\alpha \in H$. Applying $1 = (\varepsilon \otimes \mathrm{Id})(R) = \varepsilon(1)h$, we see that $h = 1$; applying $1 = (\mathrm{Id} \otimes \varepsilon)(R)$, we see that $\sum h_\alpha \varepsilon(g_\alpha) = 0$, and so all $g_\alpha \in K$. ∎

LEMMA 4.6: *Let (H, R), be a quasitriangular Hopf algebra and assume $u = 1$. Then there exist $x_i \in \operatorname{Ker} \varepsilon$ so that $f_n \cdot (1 \otimes x_2 \otimes \cdots \otimes x_n) \neq 0$ (and hence forms a k-basis for the 1-dimensional H-module $\bigwedge_R^n(H)$).*

Proof: Let $K = \operatorname{Ker} \varepsilon$. Now, $\dim K = n - 1$, and K is an H-module, since it is an ideal of H. Since $u = 1$, we may apply Corollary 2.12 to the space $V = K$ to see that $f_{n-1} \cdot K^{\otimes(n-1)}$ is a 1-dimensional H-module. Moreover, since the action of S_n is induced by Ψ, which is determined by multiplication by the R^i, we have $f_{n-1} \cdot K^{\otimes n-1} \subset K^{\otimes n-1}$. Let $x_i \in K$, so that

$$f_{n-1} \cdot (x_2 \otimes \cdots \otimes x_n) \neq 0.$$

Embed $S_{n-1} \subset S_n$ by:

$$S_{n-1} = \{\sigma \in S_n | \sigma(1) = 1\},$$

then

$$f_n = f_{n-1} + \sum_{\sigma(1) \neq 1} \operatorname{sgn}(\sigma)\sigma. \tag{26}$$

We show that $f_n \cdot (1 \otimes x_2 \otimes \cdots \otimes x_n) \neq 0$, which by Corollary 2.12 is a k-basis for $\bigwedge_R^n(H)$, and we will be done.

Consider the representation of R as in Lemma 4.5; then for any $y \in K$:

$$(1, 2) \cdot (1 \otimes y) = \sum g_\alpha y \otimes h_\alpha + y \otimes 1.$$

Similarly, if all $y_i \in K$, we have

$$(i, i+1) \cdot (y_1 \otimes \cdots \otimes 1 \otimes y_{i+1} \otimes \cdots \otimes y_n) = z_i + (y_1 \otimes \cdots \otimes y_{i+1} \otimes 1 \otimes \cdots \otimes y_n),$$

where $z_i \in K^{\otimes n}$. Thus, if $\sigma \in S_n$ and $y_i \in K$:

$$\sigma \cdot (y_1 \otimes \cdots \otimes 1 \otimes y_{i+1} \otimes \cdots \otimes y_n) = z_\sigma + (w_1 \otimes \cdots \otimes 1 \otimes \cdots \otimes w_n), \tag{27}$$

where $z_\sigma \in K^{\otimes n}$, $w_i \in K$ and 1 is in the $\sigma(i)$-th position. Thus, by (26) and (27)

$$\begin{aligned} &f_n \cdot (1 \otimes x_2 \otimes \cdots \otimes x_n) \\ &= f_{n-1} \cdot (1 \otimes x_2 \otimes \cdots \otimes x_n) + y \\ &= 1 \otimes f_{n-1} \cdot (x_2 \otimes \cdots \otimes x_n) + y \end{aligned}$$

where $y = \sum_{\sigma(1) \neq 1} \operatorname{sgn}(\sigma)\sigma(1 \otimes x_2 \otimes \cdots \otimes x_n)$.

Note that $y \in K^{\otimes n} + K \otimes 1 \otimes \cdots \otimes K + \cdots + K \otimes \cdots \otimes K \otimes 1$. Since $1 \otimes f_{n-1} \cdot (x_2 \otimes \cdots \otimes x_n)$ and y belong to different components of a direct sum, and since $f_{n-1} \cdot (x_2 \otimes \cdots \otimes x_n) \neq 0$, we deduce that

$$f_n \cdot (1 \otimes x_2 \otimes \cdots \otimes x_n) \neq 0. \qquad \blacksquare$$

We are ready to prove the main theorem of this paper. Some of the ideas are adaptations of the methods of [F-S] to this more complex situation.

THEOREM 4.7: *Let (H, R) be a triangular n-dimensional semisimple Hopf algebra over k of* $\operatorname{Char} k = 0$ *or* $\operatorname{Char} k > n$. *Let A be a quantum-commutative H-module algebra; then:*

1. *A is integral over A^H.*
2. *A is a PI ring.*
3. *If $u = 1$ then A is integral over A^H of degree n.*

Proof: We first prove (3). Let λ be an indeterminant and denote $A[\lambda]$ by E. Extend the H-action to E by defining $h \cdot \lambda = \varepsilon(h)\lambda$, all $h \in H$. Then E is a quantum-commutative H-module algebra with $E^H = A^H[\lambda]$.

Let $1_H \otimes x_2 \otimes \cdots \otimes x_n$ be as in Lemma 4.6; then by Corollary 2.16, $0 \neq 1_E \otimes f_n \cdot (1_H \otimes x_2 \otimes \cdots \otimes x_n)$ is an E-basis for $E \otimes \bigwedge_R^n(H)$. Let $m = \theta(1_E \otimes f_n \cdot (1_H \otimes x_2 \otimes \cdots \otimes x_n))$; then $f_n \cdot (E\#H)^{\otimes_E^n} = E \cdot m = m \leftarrow E$. Also note that $f_n \cdot (A\#H)^{\otimes_A^n} \subset A \cdot m$. Now let $T \in \operatorname{End}_{E\#H}(E\#H)$; then by the definition of the determinant and Theorem 2.19, $T^{\otimes n}(m) = (\det T)m$, where $\det T \in A^H[\lambda]$. In particular, let $a \in A$ and let

$$T\colon E\#H \to E\#H$$

be right multiplication by $\lambda - a$. Then $T \in \operatorname{End}_{E\#H}(E\#H)$, and λ pulls through to the left.

Viewing $H \subset E\#H$ as usual, we have:

$$\begin{aligned} &T^{\otimes n}\theta(1_E \otimes 1_H \otimes x_2 \otimes \cdots \otimes x_n) \\ &\quad = T^{\otimes n}(1 \otimes_E x_2 \otimes_E \cdots \otimes_E x_n) \\ &\quad = (\lambda - a) \otimes_E x_2(\lambda - a) \otimes_E \cdots \otimes_E x_n(\lambda - a) \\ &\quad \in (\lambda - a)(E\#H)^{\otimes_E^n}. \end{aligned}$$

By Corollary 2.17, $T^{\otimes n}$ commutes with the actions of $E\#H$ and f_n, thus by the above:

$$\begin{aligned}&T^{\otimes n}(m)\\&\quad= T^{\otimes n}(f_n \cdot \theta(1_E \otimes 1_H \otimes x_2 \otimes \cdots \otimes x_n)\\&\quad\in (\lambda - a) f_n \cdot (E\#H)^{\otimes_E^n} = (\lambda - a)E \cdot m.\end{aligned}$$

However $T^{\otimes n}(m) = \det(T)\cdot m$. Thus $\det(T) \in (\lambda - a)E$, and hence $\det(T)(a) = 0$. Since $\det(T) \in A^H[\lambda]$, we have shown that a satisfies a polynomial over A^H. Finally, since λ pulls to the left we have:

$$\begin{aligned}(\det T)\cdot m =&(\lambda - a) f_n \cdot [1 \otimes_E (\lambda x_2 - x_2 a) \otimes_E \cdots \otimes_E (\lambda x_n - x_n a)]\\=&\lambda^n f_n \cdot (1 \otimes_E x_2 \otimes_E \cdots \otimes_E x_n) + f_n \cdot Z\end{aligned}$$

where $Z \in \lambda^{n-1}(A\#H)^{\otimes_E^n} + \lambda^{n-2}(A\#H)^{\otimes_E^n} + \cdots + (A\#H)^{\otimes_E^n}$. Hence $f_n \cdot Z \in \lambda^{n-1}Am + \lambda^{n-2}Am + \cdots + Am$ and so $\det T$ is a monic polynomial of degree n. This proves (3).

1. and 2. We reduce to (3) by considering $\bar{H} = H/(u-1)H$ and then pulling back by applying known results about group gradings or actions. Specifically, $(u-1)H$ is a two-sided ideal since u is central, moreover it is a coideal of H as well, since u is group-like. Thus $(\bar{H}, \bar{R})$ is a triangular, semisimple Hopf algebra with $\bar{u} = 1$. Since H is semisimple, $u^2 = 1$. Let $G = \langle 1, u\rangle$; then G acts by automorphisms on A or equivalently A is Z_2 graded where $A = A_+ \oplus A_-$, $A_+ = A^G = \{a \in A | u \cdot a = a\}$ and $A_- = \{a \in A | u \cdot a = -a\}$. Since $(u-1)\cdot A_+ = 0$, $\bar{H}$ acts on A_+. Since A is quantum commutative with respect to (H, R), A_+ is quantum commutative with respect to $(\bar{H}, \bar{R})$. Moreover, it is obvious that $A_+^{\bar{H}} = A^H$. Applying (3) to $\bar{H}$ and A_+ we deduce that A_+ is integral over A^H of bounded degree n. Thus by [Ro1, p. 9] A_+ is a PI ring. But now, by [BC] or [M] this implies that A is a PI ring. Also, A is trivially Schelter integral over A_+ since $|G| = 2$. Applying now Theorem 4.2 to the extensions: $A^H \subset A_+ \subset A$, we deduce that A is Schelter integral over A^H. Since A^H is central in A, we deduce that A is integral over A^H. ∎

When A is assumed to be affine over k, more can be said; this is the generalization of Noether's theorem:

THEOREM 4.8: *Let (H, R) be a finite-dimensional semisimple triangular Hopf algebra over k of* Char $k = 0$. *Let A be a k-affine, quantum-commutative H-module algebra; then:*

1. *A^H is k-affine (hence Noetherian).*
2. *A is a finitely generated left and right A^H-module.*
3. *A is a left and right Noetherian PI ring.*

Proof: 1. and 2. are proved as follows: By Theorem 4.7, A is a PI ring which is integral over the central subalgebra A^H. Since A is k-affine we deduce [MR, p. 476] that A is a finitely generated A^H-module. By Artin–Tate's lemma (Theorem 4.4) applied to $k \subset A^H \subset A$ we conclude that A^H is k-affine.

3. By (1), A^H is a commutative k-affine algebra hence A^H is a Noetherian ring. Since A is a finitely generated left and right A^H-module it follows that A is a left and right Noetherian ring. It is PI by Theorem 4.7. ∎

Remark 4.9: The condition on semisimplicity of H is necessary as shown in an example in [Z]. In this example, let (H, R) be Sweedler's 4-dimensional triangular Hopf algebra over $\mathbb{C}$ (which is not semisimple), and let $A = S_R(H)$. Then A is a $\mathbb{C}$-affine quantum-commutative H-module algebra, but A^H is not $\mathbb{C}$-affine.

We ask:

QUESTION 4.10: *Is Theorem 4.7 still true omitting any characteristic assumption?*

There is already a "Noether's theorem" for Hopf algebra actions in the literature.

THEOREM 4.11 ([M2, Th. 4.3.7]): *Let A be a left Noetherian ring which is an affine k-algebra. Let H be finite-dimensional, and assume that A is an H-module algebra such that the trace $\hat{t}$: $A \mapsto A^H$ is surjective. Then A^H is k-affine.*

This result assumes that A is Noetherian, which is automatic by the Hilbert basis theorem when A is commutative. Following the referee's suggestion we ask for a kind of "quantum Hilbert basis theorem"; that is:

QUESTION 4.12: *When is an affine quantum-commutative H-module algebra necessarily Noetherian?*

Note that Theorem 4.8 offers an instance in which this question has a positive solution.

We summarize in the following some properties of the prime spectrums of A and A^H which are a direct consequence of Theorem 4.7 and [MR, p. 478].

COROLLARY 4.13: *Let (H, R), A, k be as in Theorem 4.7; then:*

1. *(Lying over) If $p \in \mathrm{Spec} A^H$ then there exists $P \in \mathrm{Spec} A$ so that $P \cap A^H = p$. There exist only a finite number of $P \in \mathrm{Spec} A$ such that $P \cap A^H = p$* [BV, 2.8].
2. *(Going up) If $p, q \in \mathrm{Spec} A^H$, with $p \subset q$ and $P \in \mathrm{Spec} A$ with $P \cap A^H = p$, then there exists $Q \in \mathrm{Spec} A$ so that $P \subset Q$ and $Q \cap A^H = p$.*
3. *(Incomparability) If $P, Q \in \mathrm{Spec} A$ with $P \subsetneq Q$, then $P \cap A^H \subsetneq Q \cap A^H$.*

Moreover, by [MR, p. 484] and the above,

COROLLARY 4.14: *If (H, R), k, A are as in Theorem 4.7 and assume A is k-affine, then:*

1. *$J(A)$, the Jacobson radical of A, is nilpotent* ([B] and Corollary 4.13).
2. *Every irreducible left (right) A-module is finite-dimensional.*
3. *A is a Jacobson ring.*
4. *A satisfies the Nullstellensatz.*

Note that for group algebras or their duals most of the applications are known. For G a finite group, if G acts on A and $|G|^{-1} \in A$, then by [Qu], A is always Schelter-integral over A^G. Moreover, by [K], if A^G is PI then so is A. If A is graded by G, then A is always Schelter-integral over A_1, a result of Bergman [Pa]. Moreover, as mentioned already in the proof of Theorem 4.7, if A_1 is PI then so is A [BC]. In either case, if A^G (respectively A_1) is central in A, then A is a PI ring integral over A^G (respectively A_1).

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